CONTRIBUTI DI RICERCA CRENOS

INVESTMENTS IN EDUCATION IN A TWO-SECTOR, RANDOM MATCHING ECONOMY

Concetta Mendolicchio

Dimitri Paolini

Tito Pietra

WORKING PAPERS

2008/09

CENTRO RICERCHE ECONOMICHE NORD SUD
(CRENoS)
UNIVERSITÀ DI CAGLIARI
UNIVERSITÀ DI SASSARI

Il CRENoS è un centro di ricerca istituito nel 1993 che fa capo alle Università di Cagliari e Sassari ed è attualmente diretto da Raffaele Paci. Il CRENoS si propone di contribuire a migliorare le conoscenze sul divario economico tra aree integrate e di fornire utili indicazioni di intervento. Particolare attenzione è dedicata al ruolo svolto dalle istituzioni, dal progresso tecnologico e dalla diffusione dell'innovazione nel processo di convergenza o divergenza tra aree economiche. Il CRENoS si propone inoltre di studiare la compatibilità fra tali processi e la salvaguardia delle risorse ambientali, sia globali sia locali.
Per svolgere la sua attività di ricerca, il CRENoS collabora con centri di ricerca e università nazionali ed internazionali; è attivo nell'organizzare conferenze ad alto contenuto scientifico, seminari e altre attività di natura formativa; tiene aggiornate una serie di banche dati e ha una sua collana di pubblicazioni.

www.crenos.it
info@crenos.it

CRENoS – CAGLIARI
VIA SAN GIORGIO 12, I-09100 CAGLIARI, ITALIA
TEL. +39-070-6756406; FAX +39-070- 6756402

CRENoS - SASSARI
VIA TORRE TONDA 34, I-07100 SASSARI, ITALIA
TEL. +39-079-2017301; FAX +39-079-2017312

Titolo: INVESTMENTS IN EDUCATION IN A TWO-SECTOR, RANDOM MATCHING ECONOMY

Investments in education in a two-sector, random matching economy

Concetta Mendolicchio[1]
IRES, Université Catholique de Louvain

and

Dimitri Paolini
DEIR, Università di Sassari

and

Tito Pietra
DSE, Università di Bologna

Louvain-La-Neuve, Belgium

Version: June 16, 2008

We consider a random matching model where heterogeneous agents endogenously choose to invest time and real resources in education. Generically in the space of the economies, there is an open interval of possible lengths of schooling such that, at at least one of the associated steady states equilibria, some agents, but not all of them, choose to invest. Regular steady state equilibria are constrained Pareto inefficient in a strong sense. The Hosios (1990) condition is neither necessary, nor sufficient, for constrained Pareto optimality. We also provide restrictions on the fundamentals, which are sufficient to guarantee that equilibria are characterized by overeducation (undereducation), and present some results on their comparative static properties.

1. INTRODUCTION

Extending the canonical Pissarides-Mortensen model (see, for instance, Pissarides (2000)), we provide a fairly general analysis of investments in human capital in a random matching model, considering two different markets, for qualified and non-qualified labor. The basic structure of the economy is simple. When they

[1]We wish to thank for their comments participants at the 16^{th} Ecole de Printemps, Aix-En-Provence 2007, and at seminars held at the Universities of Bologna, Essex, and Statale of Milan. Comments by B. Decreuse have been very helpful. B. Van der Linden kindly provided very detailed comments on a previous version of the paper, allowing us to correct many errors and obscurities. The responsibility of any remaining mistake is our own. We acknowledge the financial support of MIUR - PRIN 2006.

are born, agents can choose to invest a fixed amount of time and real resources in education to get the opportunity to enter, after graduation, the qualified labor market. Individuals choose optimally to invest, given the wage rates, the direct costs of education, and the rates of unemployment in the two markets. They are heterogeneous with respect to their potential productivities as qualified, and non-qualified, workers. Essentially, we embed a model of investments in human capital *à la* Roy (1951) (see also Willis and Rosen (1979), and Willis (1986)) in a random matching model.

In the last few decades, in particular after Becker (1964), there has been an extremely large literature on investments in human capital, looking at both the microeconomic features and their macroeconomic impact. More recently, investments in human capital have also been studied in the framework of economies with imperfect labor markets, and, particularly, in random matching models. Among many others, by Laing, Palivos and Wang (1995), Acemoglu (1996), Burdett and Smith (1996, 2002), Booth *et alii* (2005, 2006 and 2007), Charlot and Decreuse (2005, 2007), Charlot, Decreuse and Granier (2005), Becker (2006), and Tawara (2007). Most of these papers consider economies where there is a unique labor market: investments in human capital increase the number of efficiency units of labor associated with a (physical) unit of time. This set-up is adopted, for instance, in Acemoglu (1996). In his (static) model, with random matching and investments in both human and physical capital, contractual incompleteness generates a (bilateral) "hold up" problem. Underinvestment in education arises because workers anticipate that part of the productivity gains created by their irreversible investments will be captured by their future employers. The "hold up" problem plays a key role also in the model considered in Booth and Coles (2007). In a dynamic random matching model, Burdett and Smith (2002) show that there can be "low skill trap" equilibria: If few workers acquire training, firms have less incentives to create jobs. Then, poor matching prospects for the workers reduce the rate of return on skill acquisition. Tawara (2007) extends this basic approach by introducing time-to-educate. Laing, Palivos, and Wang (1995) explicitly develop a model of endogenous growth with matching frictions and investments in human capital.

Closer to our set-up are the other papers mentioned above, where there are separate labor markets for educated and uneducated workers. Becker (2006) studies the individual decision problem in an economy where education takes time and there is search while in school. Charlot and Decreuse (2005) (see also (2006)) consider an equilibrium random matching model with heterogeneous workers. They show that, when productivities and educational attainments are positively correlated, there may be a "composition effect" that induces overeducation, because the conditional expectation of the productivities of both educated and uneducated workers may decrease as more people go to school. In their model there are no opportunity costs of education, and there are very strong assumptions on the on-the-job productivities of educated and uneducated workers (they are both linear functions of "innate ability").

Our model is obviously related to these previous contributions (and mostly to the last one). In particular, as in Charlot and Decreuse (2005), we consider an economy where the investment in education is a binary choice. This is best interpreted (at least, in developed countries) as a choice about going to college. It is important to stress that the two approaches to the analysis of investments in human capital (efficiency units versus heterogeneity and binary choice) emphasize different phenomena, and may have quite different welfare implications, because the "efficiency units" approach, by assumption, ignores the, potentially important, composition effect, due to the self-selection of workers with different comparative advantages. Moreover, they have significantly different empirical implications, see, for instance, Carneiro, Heckman, and Vytlacil (2001), and Cunha and Heckman (2006)).

The class of economies that we consider presents several new features, when compared to the literature. First, workers are heterogeneous along several different dimensions: productivity on the job (and unemployment benefits, or home production, if out of work) as qualified and unqualified, and probability of graduation, if there is an investment in education. We choose not to impose any restriction on the correlations across these variables. We label individuals so that productivity if educated is strictly increasing in the "ordering" parameter θ. However, no restriction (but continuous differentiability) is imposed on the other relevant functions, and, therefore, on the expected gains in productivity due to education. Hence, we are completely agnostic regarding the existence of some intrinsic characteristics (say, "innate ability") of the individuals, which could meaningfully induce systematic correlations between their performances in different activities. In particular, we allow different agents to have comparative advantages in different jobs, which is consistent, for instance, with the results in Cunha, Heckman, and Navarro (2005). We just introduce the (very mild) assumption that, for every agent, the productivity on the job is larger when educated.

Secondly, schooling has both direct and opportunity costs. Time to educate is an important phenomenon, because, empirically, opportunity costs are, by large, the most important component of the total costs of education. For instance, in Western Europe, they are usually over 90% of the total costs and, in several countries, direct costs are actually negative, according to estimates reported in de la Fuente (2004). Therefore, it is empirically, and theoretically, worthwhile to consider them explicitly, as standard in the literature on investments in human capital (see, Becker (1964), Ben-Porath (1967), and most of the subsequent studies). In our model, opportunity costs are endogenous, determined by unemployment rates and wages.

Third, we consider two separate labor markets with different productivities and (potentially) different vacancy costs and matching functions, so that unemployment rates may vary across levels of education, which is consistent with a large empirical evidence. Variations in the unemployment rates are determined by differences in the "labor market institutions" variables and, loosely speaking, in the conditional expectation of the productivities in the two markets.

Finally, we assume that, when agents invest in education, they fail to graduate with some positive probability. This is consistent with the data: For instance, in Western Europe, the college drop-out rate varies between 15% in Ireland and 58% in Italy (See OECD (2004)). However, one may wonder why do we introduce individual risk in a model with risk-neutral agents. Apart from descriptive realism, this assumption plays an important role in the proof of the existence of steady state equilibria (SSE in the sequel), substantially simplifying the proofs. It is an open issue if existence of interior SSE could be established (under our assumptions) without this feature.

On the other hand, several, potentially important, features are not considered in the paper: First, we assume that when an agent is involved in one activity (work, or education) he cannot search for a job. We conjecture that to allow for search-on-the-job (or -in school) would not alter the main qualitative results. Moreover, we assume that labor supply and investments in physical capital are perfectly inelastic. An analysis of a (static) model *à la* Roy with labor market imperfections, elastic labor supply, and elastic investments in physical capital, is in a companion paper (Mendolicchio, Paolini, and Pietra (2008)). The main results are qualitatively similar. Finally, in our model, (at least after schooling) all the agents are in the labor force, either employed or searching for a job. As pointed out in Booth *et alii* (2005, 2006, 2007), non-participation in the labor force of a subset of agents can have important implications.

Throughout the paper, the analysis is focussed on steady states. To simplify notation and definition, we also heavily exploit the steady state condition in the definition of several features of the economy, and of the behavioral functions.

We establish five main results:

1. comparative statics properties of the aggregate demand for education;

2. generic existence of regular interior SSE for some interval of lengths of schooling;

3. generic weak constrained suboptimality of regular interior SSE.

Moreover, imposing stronger assumptions, we are able to provide:

1. a set of sufficient conditions for overeducation and undereducation;

2. some results on the comparative statics of interior SSE.

Section 2 describes the structure of the model. In our set up, the choice of education is binary, at the individual level. The measure of the set of agents investing provides us with an intuitively appealing aggregate demand for education and, in Section 3, we analyze its comparative statics properties. In Section 4, we show that, under appropriate assumptions, and generically in the space of the economies, there are interior SSE, provided that the length of schooling lies in some interval $\left(T_{\xi}, T^{\xi}\right)$. This is the "right" result. Under very general assumptions, for T sufficiently small, everyone would invest in education, because the productivity gain is bounded away from zero, while the (discounted value of the) direct and opportunity costs converge to zero. Also, for T sufficiently large, no individual would invest. For a general set of parameters, it is not possible to impose any lower bound on the "size" of the interval $\left(T_{\xi}, T^{\xi}\right)$. On the other hand, it is obvious that this interval could be, in some sense, "large". (A *caveat.* In our set up, the notion of "large" set is very problematic: interior SSE can exist only for T in some compact set, and every compact interval is small, compared with its complement in $(0, \infty)$).

In Section 5, we propose a notion of constrained Pareto efficiency, based on the idea that it should not be possible to improve upon the market allocation by simply modifying the set of people getting educated (and letting the endogenous variables adjust to their equilibrium values associated with this new set). At an interior SSE, we can have both overinvestment in education, as in Charlot and Decreuse (2005), or underinvestment, as in Acemoglu (1996). Interestingly, as in Acemoglu (1996), constrained Pareto inefficiency fails at a SSE even if the Hosios (1990) condition[2] holds.

[2] The condition is that the elasticity of the matching function is equal to the workers' weight in the bargaining process determining the wage. In the basic, one-sector, random matching model, this is a necessary and sufficient condition for constrained efficiency of SSE.

In Section 6, we move on to a somewhat simplified version of the economy and focus on the two cases of complementarity and substitutability between ability and education, providing sufficient conditions for overeducation and undereducation. For the same two cases, we also provide some results on the comparative statics of equilibria.

Throughout the paper, we interpret the existence of two separate labor markets as due to differences in the levels of education. The model could be reinterpreted, and applied to any situation where there are separate labor markets, and workers can endogenously choose to move (at a cost) across them. For instance, an additional interpretation could be related to migration phenomena.

2. THE MODEL

We start discussing the demographic structure of the economy. An agent is denoted by θ_i, where $\theta \in \Theta^0 = [\theta_\ell, \theta_h]$ describes his innate characteristics, while $i \in [0,1]$ denotes his "name". As usual, $\mu(A)$ is the Lebesgue measure of any set $A \subset \mathbb{R}^M$, for some M. We endow Θ^0 and $[0,1]$ with the Lebesgue measure, normalized so that $\mu\left(\Theta^0\right) = \mu([0,1]) = 1$, and $\Theta = \Theta^0 \times [0,1]$ with the product measure.

At each instant t, for each θ, a subset of measure γ of agents dies and is replaced by an interval of type θ agents with the same measure (and the same "names"). We need that, for each θ_i and t, the event "death of agent i" at t (conditional on his being alive at each $\tau < t$) has probability γ, and that, (at least) almost surely, the event realizes for a set of agents with measure γ. It is well-known (see, Judd (1985), and Feldman and Gilles (1985)) that the standard version of the "law of large numbers" cannot hold in our set-up. In the last few years (also in connection with random matching models), there has been a large literature studying this issue[3]. In our model, the simplest solution is *not* to assume that the realizations of the random variables "death of θ_i" are independent over i. On the contrary, following a suggestion in Feldman and Gilles (1985) (and in Alòs-Ferrer (1999)), we assume that, at each t, there is a realization of a random variable $\widetilde{\omega}$, uniformly distributed on $(0,1]$, determining the state of the world for each agent θ_i. Define the random variable

$$\Gamma_i\left(\widetilde{\omega}, t\right) = \left\{ \begin{array}{ll} \text{"death"} & \text{if } i \in (\max\left\{0, \widetilde{\omega} - \gamma\right\}, \widetilde{\omega}] \cup [1 - \min\left\{0, \widetilde{\omega} - \gamma\right\}, 1] \\ \text{"non death"} & \text{otherwise} \end{array} \right\}.$$

Evidently, at each t, a set of agents of measure γ actually dies, and, from the individual viewpoint, the probability of dying at t (if alive at each $\tau < t$) is γ, as required.

At each t, if agent θ_i dies, he is replaced by his own clone, so that the distribution of the agents is stationary. This assumption is common in the literature. At each t, there are three sets of individuals: qualified workers, denoted by L_t^e, unqualified workers, L_t^{ne}, and students. The labor force is $L_t = L_t^e + L_{t.}^{ne}$

At birth, each individual is uneducated, denoted by a superscript $k = ne$. By spending a period of fixed length T in education, and paying instantaneous direct costs described by a function $c(\theta)$, he becomes educated (or "qualified"), denoted by a superscript $k = e$, with probability $\alpha(\theta)$. To simplify, for each θ, the individual

[3] Among many others, Al-Najjar (2004), Alòs-Ferrer (1999), Duffie and Sun (2007), and Sun (2007).

random variable success/failure in education realizes at the end of schooling. This is a strong assumption (and generally inaccurate from the descriptive viewpoint). As long as the probability of failure is exogenous, our results are robust to more realistic descriptions of the phenomenon. For instance, we could introduce an exogenous stochastic process for the failure rate over the education period. We need that, for each θ, just a fraction $\alpha(\theta)$ of the agents actually graduates and that, for each individual θ_i, the probability of graduation is $\alpha(\theta)$. To obtain this result, we adopt the same construction introduced above, using a $\theta-$specific random variable $\widetilde{\varkappa}_\theta$, uniformly distributed on $(0,1]$. As above, we can then construct a $\theta-$specific random variable $g_\theta\left(\widetilde{\varkappa}\right)$, selecting the subset (of measure $\alpha(\theta)$) of agents of type θ who actually graduate. In the sequel, for notational convenience, we will assume that, afterwards, we rename the agents, so that, for each θ, the students who actually graduate have $i \in \left[0, e^{-\gamma T}\alpha\left(\theta\right)\right]$.

Productivities on the job and in home production (and/or unemployment benefits) depend upon innate characteristics and the level of education. If educated and working as such, a worker has output $f^e\left(\theta\right)$ (or, if unemployed, home production $b^e\left(\theta\right)$). Otherwise, he produces $f^{ne}\left(\theta\right)$ (or $b^{ne}\left(\theta\right)$). We assume that, after graduation, workers cannot simultaneously search for a job in both markets. Hence, to simplify notation (and, given that education is costly, without any loss of generality, at a SSE), educated individuals only look for a job on market e. The functions (f, b) are time-invariant. This implies that human capital does not depreciate, and that there is no learning-by-doing. Again, more realistic assumptions would not affect the results, as long as these phenomena are described by exogenous (maybe stochastic) processes.

More formally, instantaneous output is given by a function $f : [\theta_\ell, \theta_h] \times \{e, ne\} \to \mathbb{R}_+$. Home production by a function $b : [\theta_\ell, \theta_h] \times \{e, ne\} \to \mathbb{R}$. We assume that all the relevant functions are at least C^3 on some open neighborhood $(\theta_\ell - \varepsilon, \theta_h + \varepsilon)$, that individuals are more productive when educated (i.e., $f^e(\theta) > f^{ne}(\theta)$, and $b^e(\theta) \geq b^{ne}(\theta)$, for each θ), and that productivity on-the-job is larger than home production. These are fairly natural assumptions. Moreover, we relabel individuals so that $f^e\left(\theta\right)$ is strictly increasing. We do not impose any additional monotonicity restriction on the other functions. To summarize,

Assumption 1:

- For each k, $f^k\left(\theta\right), b^k\left(\theta\right), \alpha\left(\theta\right), c(\theta) \in C^3$ on $\left(\theta_\ell - \varepsilon, \theta_h + \varepsilon\right)$, for some $\varepsilon > 0$;
- $f^e\left(\theta\right)$ is strictly monotonically increasing in θ on the set $\left(\theta_\ell - \varepsilon, \theta_h + \varepsilon\right)$;
- for each k, $\frac{1}{\delta} \geq \left(f^k(\theta) - b^k(\theta)\right) \geq \delta$, for some $\delta > 0$, $1 > \alpha\left(\theta\right) > 0$ and $c\left(\theta\right) > 0$, for each θ;
- for each θ, $f^e(\theta) > f^{ne}(\theta)$ and $b^e(\theta) \geq b^{ne}(\theta)$.

Let $\mathcal{F}$ be the space of functions (f, b, c, α) satisfying Assumption 1.

Agents are endowed with a Von Neumann - Morgenstern utility function and are risk-neutral [4]. Given the assumption of risk-neutrality, there is no essential loss of

[4] With individual risk (induced by the possibility of failing to graduate and of death), this is a stronger assumption than usual. Without individual risk, to impose as objective the maximization of discounted expected income (instead of expected utility of income) can be justified making appeal to market completeness. Here, we would need market completeness with respect to individual

generality in assuming that each agent knows his own θ (i.e., his productivities on-the-job and in home production, his direct costs of education and his probability of graduation). A firm, after the match, observes the value θ of the agent it is matched with (i.e., it observes $f^k(\theta)$ and $b^k(\theta)$, the only relevant variables from its viewpoint).

A final remark on notation: We will often take integrals of some function of θ, say $f^e(\theta)$, over some subset of L_t, say L_t^e. To avoid confusion, we will use the notation $\int_{L_t^e} f^e(\vartheta)\,d\vartheta$ and use, for instance, $\partial\left(\int_{L_t^e} f^e(\vartheta)\,d\vartheta\right)/\partial\theta_m$ to denote the derivative with respect to the bound θ_m of (one of the) intervals of integration (assuming that this is meaningful). Also to simplify the notation, the same function, say, $G(.)$, will be sometime written $G(\theta,\phi,T;\xi)$, sometime, for instance, $G(\theta,\phi;\xi)$. This simply means that the ignored variable, here T, is taken as given. Moreover, we are only interested in steady states, and we will omit the index t, whenever it is possible.

2.1. Workers

Existence of a continuum of agents with identical θ is only introduced to guarantee that a measure $e^{-\gamma T}\alpha(\theta) < e^{-\gamma T}$ of $\theta-$individuals will actually graduate. In the sequel, whenever possible, we will omit the subscript "i". Moreover, we will leave implicit, most of the time, the "almost surely" qualification of many of our statements.

After birth (or after attending school) agents enter the labor market, searching for a job. An agent, if active on labor market k, receives job offers according to a Poisson process whose arrival rate π^k is endogenously given by a (possibly, $k-$dependent) matching function.

Let's define

- $U^k(\theta)$ = expected life-time utility of search of a $\theta-$agent with education k;
- $V^k(\theta)$ = expected life-time utility of a match of a $\theta-$agent with education k.

Also, let r' be the (type- and education-invariant) interest rate, and $w^k(\theta)$ be the wage rate of a $\theta-$agent, if k. For notational convenience, let $(\gamma + r') = r$.

When a match obtains, $V^k(\theta) = \frac{w^k(\theta)}{r}$. It is straightforward to check that, when unemployed, the discounted, expected utility is $U^k(\theta) = \frac{\pi^k w^k(\theta) + r b^k(\theta)}{r(r+\pi^k)}$.

Assume that capital markets are perfect and, without any essential loss of generality, let $c(\theta)$ be time-invariant. The discounted, expected utility of education of agent θ, $H(\theta)$, is then

$$H(\theta,\pi,T) = \frac{e^{-rT}}{r}\left[\alpha(\theta)U^e(\theta) + (1-\alpha(\theta))U^{ne}(\theta)\right] - \frac{\left(1-e^{-rT}\right)}{r}c(\theta),$$

i.e., if θ_i chooses to invest in education, he bears the direct costs up to period T. Then, if he graduates (which happens with probability $\alpha(\theta)$), he enters the labor market for educated workers. Otherwise (with probability $(1-\alpha(\theta))$, he enters the other market.

risk, a much less compelling assumption, even less so if we would take into account the possibility of moral hazard problems. On the other hand, abstracting from moral hazard issues, the main results of the paper could be established for risk-averse individuals, provided that the degree of risk aversion is sufficiently small.

Evidently, an agent invests in education only if $H(\theta) \geq U^{ne}(\theta)$. Solving $H(\theta) - U^{ne}(\theta) = 0$, and using continuity of the maps, we can partition Θ into two (measurable) subsets of individuals, the set of agents choosing to invest in education, Θ^e, and its complement, Θ^{ne}. For the sake of concreteness, let's assume that all the indifferent agents choose to invest. Hence, by our tie-breaking rule, rearranging variables, and multiplying by re^{rt}, an agent of type θ chooses to get educated if and only if

$$G(\theta, \pi, T) = \alpha(\theta)\left(U^e(\theta) - U^{ne}(\theta)\right) + \left(1 - e^{rT}\right)\left(c(\theta) + U^{ne}(\theta)\right) \geq 0. \qquad (1)$$

The function $G(\theta, \pi, T) / re^{rt}$ gives us the discounted, expected value of the investment in education: the discounted expected value of the gain from education, minus its direct and opportunity costs.

Consider the cohort born at time t, and define the following sets:

- $\Theta_t^e = \{\theta_i \in \Theta | G(\theta_i, .) \geq 0\}$,
- $\Theta_t^{0e} = \left\{\theta \in \Theta^0 | G(\theta, .) \geq 0\right\}$,
- $\Theta_t^{ne} = \{\theta_i \in \Theta | G(\theta_i, .) < 0\}$,
- $\Theta_t^{0ne} = \left\{\theta_i \in \Theta^0 | G(\theta, .) < 0\right\}$,

and, at time $(t + T)$,

- $\Theta_t^{e\alpha} = \left\{\theta_i \in \Theta | G(\theta_i, .) \geq 0 \text{ and } i \leq e^{-\gamma T}\alpha(\theta)\right\}$.

Modulo a relabelling of the agents, the last set gives one (equivalent) representation of the set of individuals who actually graduate.

Given a sequence $\left\{\Theta_t^{0e}\right\}_{t=0}^{\infty}$, with $\Theta_t^{0e} = \Theta^{0e}$, each t, the stationary sets (L^e, L^{ne}) have measures

$$\mu(L^e) = \mu(\Theta^{e\alpha}) \qquad \text{and} \qquad \mu(L^{ne}) = \mu(\Theta^{ne}) + e^{-\gamma T}\mu(\Theta^e) - \mu(\Theta^{e\alpha}),$$

or

$$\begin{aligned} \mu(L^e) &= e^{-\gamma T}\int_{\Omega^{0e}} \alpha(\vartheta)\, d\vartheta \qquad (2) \\ \mu(L^{ne}) &= \int_{\Omega^{0ne}} d\vartheta + e^{-\gamma T}\int_{\Omega^{0e}} (1 - \alpha(\vartheta))\, d\vartheta. \end{aligned}$$

The stationary measure of people in school is $\mu(S) = \left(1 - e^{-\gamma T}\right)\mu(\Theta^e)$. Evidently, $\mu(S) + \mu(L^{ne}) + \mu(L^e) = 1$. Finally, bear in mind that continuity of $G(\theta, \pi, T)$ implies that, given any (π, T) the set Θ_t^{0e} is the union of a finite collection of closed intervals and (possibly) isolated points.

2.2. Firms

Each firm is endowed with a technology allowing it to use one unit of labor to produce a quantity of homogeneous output. We abstract from investments in physical capital. While workers can move from the skilled labor market to the unskilled one, firms cannot. Firms (potentially) active in each one of the two markets are identical and there is an unlimited number of potential entrants in each one of them. Given that, at the equilibrium, expected profits are nil, to restrict firms to be active only in one market does not entail any loss of generality. We can think of the two sectors as defined by different technologies used to produce the same physical commodity, or (more plausibly) as sectors producing different physical commodities. If this is the case, in this partial equilibrium setting, we take as given their two prices, and, by choosing appropriately the units of measurement, we set both equal to 1, so that we can conveniently drop them from the notation.

The set up of the demand side of each one of the two labor markets is standard. Under perfect capital markets, firms have the same discount factor r'. To open a vacancy in labor market k entails a fix, instantaneous cost, $v^k, k = ne, e$, satisfying $v = (v^{ne}, v^e) >> 0$. Usually, they are interpreted as advertising and recruitment costs. Also, remember that $(1-\gamma)$ is the rate of survival of a match at t, that can be terminated because the worker drops out of the labor force. As in the previous section, we replace $(\gamma + r')$ with r. Finally, for each firm active on market k, matches are governed by a Poisson process with arrival rate q^k (endogenously determined by the matching function).

Let V^k be the expected value of a vacancy in market k. Let $J^k(\theta)$ be the expected value of a match with a worker θ, and J^k be its expected value, conditional on L^k. Then, in the time interval Δ, the expected gain from opening a vacancy is $V^k = -v^k\Delta + \left[q^k\Delta J^k + \left(1 - q^k\Delta\right)V^k\right]$.

Assuming perfect competition, vacancies are created up to the point where $V^k = 0$, and, therefore, at a SSE,

$$J^k = \frac{v^k}{q^k}, \tag{3}$$

i.e., the discounted conditional expectation of the value of the flow of expected gains from a match is equal to the cumulated costs of maintaining a vacancy.

The flow of profits induced by a vacancy filled by a θ worker is $\left(f^k(\theta) - w^k(\theta)\right)$, until he drops out of the match. Hence, the expected value (conditional on L^k) of a filled vacancy is

$$J^k = \frac{\int_{L^k}\left(f^k(\vartheta) - w^k(\vartheta)\right)d\vartheta}{r\mu\left(L^k\right)}. \tag{4}$$

Substituting into (3), we obtain the zero expected profit condition

$$\frac{\int_{L^k}\left(f^k(\vartheta) - w^k(\vartheta)\right)d\vartheta}{r\mu\left(L^k\right)} = \frac{v^k}{q^k}. \tag{5}$$

2.3. Bargaining

After each match, the shares of output of the worker and the firm are determined by a bargaining process, taking place after the type of the worker is revealed (equivalently, the wage is output - i.e., worker's type - contingent). The firm and the worker bargain over their shares of total output adopting the Nash bargaining solution criterion, with exogenous weights respectively $(1-\beta)$ and β. The outside options are, respectively, $U^k(\theta)$, for a worker of type θ and qualification k, and $V^k = 0$ for each firm, by the assumption of perfect competition. The output shares are obtained solving

$$\max\left(V^k(\theta) - U^k(\theta)\right)^\beta \left(J^k(\theta)\right)^{1-\beta} \equiv \left(\frac{w^k(\theta) - rU^k(\theta)}{r}\right)^\beta \left(\frac{f^k(\theta) - w^k(\theta)}{r}\right)^{1-\beta}.$$

At a SSE, $V^k(\theta) = 0$. Hence, each firm always hires the first worker it meets. Solving, we obtain the wage of a $\theta-$worker,

$$w^k\left(\theta, \pi^k\right) = \beta \frac{\left(r + \pi^k\right) f^k(\theta)}{r + \beta\pi^k} + (1-\beta) \frac{rb^k(\theta)}{r + \beta\pi^k}. \tag{6}$$

2.4. Matching and unemployment

At instant t, on market k, the measure of unemployed agents is $u_t^k \mu\left(L_t^k\right)$, $o_t^k \mu\left(L_t^k\right)$ is the measure of vacant jobs ("openings"), expressed in terms of the measure of the labor force of type k. At each t, a measure

$$m_t^k = m^k(u_t^k \mu\left(L_t^k\right), o_t^k \mu\left(L_t^k\right)).$$

of matches take place. As usual, we adopt the following

Assumption 2: $m^k(u_t^k \mu\left(L_t^k\right), o_t^k \mu\left(L_t^k\right)) \in C^3$, satisfies $\nabla m^k >> 0$, is concave, homogeneous of degree 1 in $\left(u_t^k \mu\left(L_t^k\right), o_t^k \mu\left(L_t^k\right)\right)$ (constant returns to scale) and satisfies the Inada's condition.

Using $q_t^k o_t^k \mu\left(L_t^k\right) = m^k(u_t^k \mu\left(L_t^k\right), o_t^k \mu\left(L_t^k\right))$, defining as $\phi_t^k \equiv \frac{o_t^k}{u_t^k}$ the "market tightness" variables, and exploiting homogeneity of degree 1, we obtain

$$q_t^k = m^k(\frac{u_t^k}{o_t^k}, 1) \equiv q^k(\phi_t^k),$$

and

$$\pi_t^k = m(1, \frac{o_t^k}{u_t^k},) = \phi_t^k q^k(\phi_t^k) \equiv \pi^k\left(\phi_t^k\right),$$

where π_t^k $\left(q_t^k\right)$ is the arrival rate, at t, of the Poisson process governing matches for workers (firms) in sector k. Hence, the measure of the flow of workers into employment in a time interval Δ is $\pi^k(\phi_t^k) u_t^k \mu\left(L_t^k\right) \Delta$. As usual, for each k, $\frac{\partial q^k}{\partial \phi^k} < 0$ and $\frac{\partial \pi^k}{\partial \phi^k} > 0$. Also, let $\eta_{q^k\left(\phi^k\right)} = \frac{\phi^k}{q^k\left(\phi^k\right)} \frac{\partial q^k}{\partial \phi^k} \in (-1, 0)$ and $\eta_{\pi^k}(\phi^k) = \frac{\phi^k}{\pi^k\left(\phi^k\right)} \frac{\partial \pi^k}{\partial \phi^k} = 1 + \eta_{q^k\left(\phi^k\right)}$.

In the time-interval of length Δ, the set of unemployed is affected by the flows of individuals dropping out of the labor force, $\left(\gamma u_t^k \mu\left(L_t^k\right) \Delta\right)$, or getting a job, $\left(\pi^k(\phi_t^k) u_t^k \mu\left(L_t^k\right) \Delta\right)$. The measure of the flow of individuals into unemployment is due to the new workers replacing the fraction of workers leaving the market. For the educated workers, at each t, it is given by the inflow of people who, at $(t-T)$, had chosen to get into education, and have both survived and succeeded, i.e., $\gamma\mu\left(\Theta_{t-T}^{e\alpha}\right)$. Therefore,

$$\frac{\partial u_t^e \mu\left(L_t^e\right)}{\partial t} = -\gamma u_t^e \mu\left(L_t^e\right) - \pi^e\left(\phi_t^e\right) u_t^e \mu\left(L_t^e\right) + \gamma\mu\left(\Theta_{t-T}^{e\alpha}\right)$$

Using (2) above, and setting $\frac{\partial u_t^e \mu\left(L_t^e\right)}{\partial t} = 0$, we obtain that the steady state rate of unemployment for the educated agents is

$$u^{e*} = \frac{\gamma}{\gamma + \pi^e\left(\phi^e\right)}.$$

Similarly, the measure of unemployed, uneducated people is affected by the inflow of people born at $(t-T)$ who chose to get into education, and have both survived and failed, and by the set of people born at t who choose not to get educated. Hence,

$$\begin{aligned}\frac{\partial u_t^{ne}\mu\left(L_t^{ne}\right)}{\partial t} &= -\gamma u_t^{ne}\mu\left(L_t^{ne}\right)-\pi^{ne}\left(\phi_t^{ne}\right)u_t^{ne}\mu\left(L_t^{ne}\right) \\ &\quad +\gamma\left(\mu\left(\Theta_t^{ne}\right)+\left(e^{-\gamma T}\mu\left(\Theta_{t-T}^{e}\right)-\mu\left(\Theta_{t-T}^{e\alpha}\right)\right)\right.\end{aligned}$$

Hence, using again (2) above, at a steady state,

$$u^{ne*}=\frac{\gamma}{\gamma+\pi^{ne}\left(\phi^{ne}\right)}. \tag{7}$$

Bear in mind that u_t^k denotes the rate of unemployment in labor market k, at time t, while $u_t^k\mu\left(L_t^k\right)$ is the measure of unemployed workers.

2.5. The space of the economies

Most of our results hold for an open, dense[5] subset of economies. Hence, we need to define precisely the space of the economies, and to endow it with a topological structure. The parameters defining the economy are: vacancy costs, $v\in\mathbb{R}_{++}^2$, a pair of matching functions satisfying Assumption 2 above, m^k, and a vector (f,b,c,α) of production, benefits, direct costs, and probability of success in education functions, satisfying Assumption 1 above. An economy is

$$\xi=(v,m,f,b,c,\alpha)\in\Xi.$$

We endow $\mathbb{R}_{++}$ with the Euclidean topology, all the functional spaces with the C^3 compact-open topology, $\mathbb{R}_{++}^2$, and Ξ with the product topology. It is well known that Ξ is a metric space. The distance, for instance, between m and m' depends upon the distance (on compacta) between the values of the functions, m^e (and m^{ne}) and $m^{e\prime}$ (and $m^{ne\prime}$), and of their first, second, and third order derivatives. The notions of convergence and openness of sets are defined accordingly.

3. COMPARATIVE STATICS OF THE AGGREGATE DEMAND FUNCTION FOR EDUCATION

By replacing (6) into (1), we obtain

$$\begin{aligned}G(.) &= \alpha(\theta)\left[\frac{\pi^e\left(\phi^e\right)\beta f^e(\theta)+rb^e(\theta)}{r+\beta\pi^e\left(\phi^e\right)}-\frac{\pi^{ne}\left(\phi^{ne}\right)\beta f^{ne}(\theta)+rb^{ne}(\theta)}{r+\beta\pi^{ne}\left(\phi^{ne}\right)}\right] \\ &\quad +\left(1-e^{rT}\right)\left(c(\theta)+\frac{\pi^{ne}\left(\phi^{ne}\right)\beta f^{ne}(\theta)+rb^{ne}(\theta)}{r+\beta\pi^{ne}\left(\phi^{ne}\right)}\right).\end{aligned} \tag{8}$$

Evaluated at a stationary sequence $\{\phi_t\}_{t=\tau}^{\infty}$, $\phi_t=\phi$, for each t, the measure of the set agents investing in education, $\mu\left(\Theta^{0e}(\phi;\xi)\right)$, is implicitly defined by the condition $G(\theta,\phi;\xi)\geq 0$. It can be interpreted as the aggregate demand function for education. It is easy to see that $\mu\left(\Theta^{0e}(\phi;\xi)\right)$ may fail to be differentiable,

[5] B $\subset A$ is a dense subset of A, if, for each $x\in A$, and each open ball centered on $x, V_\epsilon(x)$, $B\cap V_\epsilon(x)\neq\emptyset$.

and even to be continuous. Discontinuities are due to the possibility that there is some $\theta \in G^{-1}(0)$ which is a degenerate local extremum (as usual, $G^{-1}(0)$ denotes the set of values of θ such that $G(\theta, \phi; \xi) = 0$, for a given pair $(\phi; \xi)$). Lack of differentiability is due to the possibility that there is some $\theta \in G^{-1}(0)$ which is a (non-degenerate) local extremum, or an inflexion point. Next, we show that, for a generic subset of parameters, $\mu\left(\Theta^{0e}(\phi; \xi)\right)$ is a continuous function, and we establish some comparative statics properties of the aggregate excess demand functions for economies in this generic set. For technical reasons, Lemma 1 holds at each ϕ contained in some compact subset of $\mathbb{R}^2_{++}$. This restriction is fairly innocuous, because, as we will establish later on, given any economy ξ, and length of schooling T, the SSE values of ϕ are in fact contained in some compact subset of $\mathbb{R}^2_{++}$.

Lemma 1. *Given any sequence $\{\phi_t\}_{t=\tau}^{\infty}$, with $\phi_t = \phi$, for each t, and $\phi \in F_\xi$, a compact subset of $\mathbb{R}^2_{++}$, there is an open, dense subset of economies $\Xi' \subset \Xi$ such that, for each $\xi \in \Xi'$, $\mu\left(\Theta^{0e}(\phi; \xi)\right)$ is a continuous function at each $\phi \in F_\xi$.*

Proof. See Appendix 2. ∎

Consider the one-dimensional parameterization of the various functions, defined, for instance, by $\alpha(\theta, a) = (1 + a)\,\alpha(\theta)$, and let

$$\nabla_\alpha \mu(\phi; \xi) \equiv \left[\mu\left(\Theta^{0e}\left(\phi; \alpha(\theta, a), \xi^{\backslash}\right)\right) - \mu\left(\Theta^{0e}\left(\phi; \alpha(\theta, 0), \xi^{\backslash}\right)\right)\right]$$

be the change in the measure of the set $\Theta^{0e}(\phi; \xi)$ due to a change of $\alpha(\theta)$, *coeteris paribus*. The (intuitively appealing) comparative statics properties of the (stationary) demand of education function are reported in Proposition 1.

Proposition 1. *Given any sequence $\{\phi_t\}_{t=\tau}^{\infty}$, with $\phi_t = \phi$, for each t, and $\phi \in F_\xi$, a compact subset of $\mathbb{R}^2_{++}$, there is an open, dense subset of economies $\Xi' \subset \Xi$ such that, for each economy $\xi \in \Xi'$, the continuous function $\mu\left(\Theta^{0e}(\phi; \xi)\right)$ satisfies the following sign restrictions:*

$$\begin{bmatrix} \nabla_{\phi^e}\mu & \nabla_{\phi^{ne}}\mu & \nabla_\alpha\mu & \nabla_{f^e}\mu & \nabla_{f^{ne}}\mu & \nabla_{b^e}\mu & \nabla_{b^{ne}}\mu & \nabla_c\mu \\ \geq 0 & \leq 0 & \geq 0 & \geq 0 & \leq 0 & \geq 0 & \leq 0 & \leq 0 \end{bmatrix}.$$

Moreover, assume that the change in $b^k(\theta, a_b)$ is k-invariant, then, $\nabla_b \mu \leq 0$ if $\pi(\phi^e) > \pi(\phi^{ne})$.

Proof. See Appendix 2. ∎

Remark 1. Proposition 1 is, actually, independent of Lemma 1. The restriction to the generic set $\Xi' \subset \Xi$ is only required for the continuity property. The sign restrictions hold even for non-continuous functions $\mu\left(\Theta^{0e}(\phi; \xi)\right)$. On the other hand, stationarity of the sequence $\{\phi_t\}_{t=\tau}^{\infty}$ is obviously crucial for our argument of proof to hold. A similar result should hold for any sequence in some compact set. However, the details of the proof would be different.

4. EQUILIBRIUM

As usual, we define SSE in terms of the pair of "market tightness" variables $\phi = (\phi^e, \phi^{ne})$.

Bear in mind that, given that $\alpha(\theta) \in (0, 1)$, for each θ, the set $L^{ne}(\phi)$ is always non-empty. On the contrary, it may very well be that $L^e(\phi) = \emptyset$. For such a ϕ, in the definition of equilibrium, we impose a weak restriction on the set of allowable (ex-ante) expected profits. This restriction leaves a lot of freedom in constructing equilibria with $L^e(\phi) = \emptyset$, probably too much freedom. However, in the sequel, we will not consider this sort of trivial equilibria.

Definition 1. A SSE is a pair (ϕ^e, ϕ^{ne}), and associated $(w^e(\theta), w^{ne}(\theta))$ and $(L^e(\phi), L^{ne}(\phi))$, such that, for each k:

i. $\frac{\int_{L^k(\phi)}\left(f^k(\vartheta)-w^k(\vartheta)\right)d\vartheta}{r\mu(L^k(\phi))} \leq \frac{v^k}{q^k\left(\phi^k\right)}$, with equality if $o^k \neq 0$,

ii. $w^k(\theta) = \beta\frac{\left(r+\pi^k\left(\phi^k\right)\right)f^k(\theta)}{r+\beta\pi^k\left(\phi^k\right)} + (1-\beta)\frac{rb^k(\theta)}{r+\beta\pi^k\left(\phi^k\right)}$,

iii. $L^e = \left\{\theta_i \in \Theta | G(\theta) \geq 0 \text{ and } i \leq e^{-\gamma T}\alpha(\theta)\right\}$, and $L^{ne} = \{\theta \in \Theta | G(\theta) < 0\} \cup \left\{\theta_i \in \Theta | G(\theta) \geq 0 \text{ and } i \in \left[e^{-\gamma T}\alpha(\theta), e^{-\gamma T}\right]\right\}$.

If $L^e(\phi) = \emptyset$, we define

$$\frac{\int_{L^e(\phi)}(f^e(\vartheta) - w^e(\vartheta))\,d\vartheta}{r\mu(L^e(\phi))} = \lim_{n\to\infty}\frac{\int_{L^{en}}(f^e(\vartheta) - w^e(\vartheta))\,d\vartheta}{r\mu(L^{en})},$$

for some sequence $\{L^{en}\}_{n=1}^{n=\infty}$ with $\mu(L^{en}) \neq 0$, for each n, and $\lim_{n\to\infty}\mu(L^{en}) \to 0$.

A SSE is *interior* if and only if $\Theta^e(\phi) \neq \emptyset$ and $\Theta^e(\phi) \neq \Theta$.

Replacing (ii) into (i), we can rewrite the non-positive profits conditions as

$$\Phi^k(\phi;\xi) \equiv \frac{1-\beta}{r+\beta\pi^k(\phi^k)}\frac{\int_{L^k(\phi)}\left(f^k(\vartheta) - b^k(\vartheta)\right)d\vartheta}{\mu\left(L^k(\phi)\right)} - \frac{v^k}{q^k(\phi^k)} \leq 0, \text{ for each } k. \quad (9)$$

By definition, ϕ^* is an *interior* SSE, given T^*, if and only if $\Phi(\phi^*;\xi) = 0$, and $\Theta^e(\phi^*;\xi) \neq \Theta$, $\Theta^e(\phi^*;\xi) \neq \emptyset$.

In establishing existence of SSE, the main difficulty is that, for T sufficiently large, there is always a trivial equilibrium where no one invests in education[6]. Indeed, let ϕ^{ne° be the (unique) SSE of the associated economy with no investment in human capital. For T sufficiently large, this can be supported as a SSE when there are "sufficiently pessimistic" expectations. For instance, choose any ϕ^{e° such that $\left(\min_\theta \frac{(1-\beta)(f^e(\theta)-b^e(\theta))}{r+\beta\pi^e(\phi^{e^\circ})} - \frac{v^e}{q^e(\phi^{e^\circ})}\right) < 0$. Then, for firms with expectations given by $\overset{\circ}{\phi}$ and $L^e = \arg\min_\theta(f^e(\theta) - b^e(\theta))$, it is optimal not to create a vacancy. Consider now the value of the map $G(\theta, \overset{\circ}{\phi}, T)$. For T sufficiently large, $G(\theta, \overset{\circ}{\phi}, T) < 0$, for each θ. Hence, no worker has any incentive to invest in education. Therefore, $\Phi^e(\overset{\circ}{\phi}, T; \xi)$ is not well-defined, and it is easy to construct a sequence with the properties required in Definition 1, so that $\overset{\circ}{\phi}$ is a SSE. This trivial SSE may exists even if the same economy has an interior SSE, too. What matters is that, for T sufficiently large, this is the *unique* SSE (modulo the indeterminacy of the value of ϕ^{e°). On the other hand, under fairly general assumptions, for T sufficiently small, there is a SSE where all the agents invest in education. Therefore, the best we can look for is the existence of an interval $\left(T_\xi, T^\xi\right)$ such that, for each $T \in \left(T_\xi, T^\xi\right)$, there is an interior equilibrium.

[6] Trivial "autarkic" equilibria with no vacancies and no labor force in one sector (or in both) also exist, as usual in random matching models. The difference is that, in our economy, for T sufficiently large, there are no SSE with $\Theta^e \neq \emptyset$.

We establish two distinct existence results. The first one is that, given T, for a generic set of economies, a SSE exists. This result is only generic essentially because of the possible lack of continuity of the map $\mu(L^e(\phi;\xi))$, already discussed in Section 3. However, once continuity of $\Phi^k(\phi;\xi)$, $k = e, ne$, is established, existence of a SSE follows by a routine fixed point argument.

THEOREM 1. *Under assumptions 1-2, given* T*, for each* $\xi \in \Xi^*$*, an open, dense subset* $\Xi^* \subset \Xi$*, there is a SSE* ϕ^*.

Proof. See Appendix 3. ∎

Unfortunately, a fixed point argument does not seem to suffice to establish existence of *interior* SSE. Hence, we need a different approach to establish this second (and more interesting) result.

The basic logic of the proof is straightforward. Given that the details are somewhat pesky, we start presenting an outline. Let T^* be the largest value of T such that there is a SSE ϕ^* with $\Theta^{0e}(\phi^*, T;\xi) = \Theta^0$. As we will establish later on, T^* exists. If $\Phi(\phi, T;\xi)$ were C^1 at T^*, and $\det D_\phi \Phi(\phi, T;\xi) \neq 0$, existence of interior SSE, for each $T > T^*$ in some open neighborhood of T^*, would follow immediately. Indeed, using the implicit function theorem (from now on, IFT), we could construct a map $\phi(T)$ such that $\Phi(\phi(T), T;\xi) = 0$, for T close to T^*. Existence of interior SSE would follow immediately, because, by construction, at $T > T^*$, $\Theta^e(\phi(T), T;\xi) \neq \Theta$, and, by continuity, $\Theta^e(\phi(T), T;\xi) \neq \emptyset$, for T close to T^*. The difficulty is that, at (ϕ^*, T^*), $\Phi(\phi^*, T^*;\xi)$ is necessarily non differentiable, because each $\theta^* \in G^{-1}(0)$ is either on the boundary of $[\theta_\ell, \theta_h]$ or, worst, an interior minimum of $G(\theta, \phi, T;\xi)$, so that $\frac{\partial G}{\partial \theta} = 0$. In both cases, $\Phi(\phi, T;\xi)$ fails to be differentiable at (ϕ^*, T^*). However, for T sufficiently close to T^*, and an appropriate perturbation of the parameters of the economy, there is a SSE $\phi(T)$ such that $\Phi(\phi(T), T;\xi)$ is C^1 and $\det D_\phi \Phi \neq 0$. Existence of this SSE follows by continuity of the maps $G(.)$ and $\Phi(.)$ at (ϕ^*, T^*). This is a much weaker condition than differentiability (plus $\frac{\partial G}{\partial \theta} \neq 0$). Still, it is not necessarily satisfied. However, for a generic set of economies, it can be established along the lines of the proof of Lemma 1.

Our existence results are summarized in Theorem 2. The proof rests on the fact that there is a SSE where everyone invests in education, for T sufficiently small. It is easy to find assumptions on the primitives such that this property holds. Lemma 2 describes *one* such a set.

LEMMA 2. *Under the maintained assumptions, let* $\frac{\int_{\Theta^0} \alpha(\vartheta)(f^e(\vartheta) - b^e(\vartheta))d\vartheta}{\int_{\Theta^0} \alpha(\vartheta)d\vartheta} > \frac{\int_{\Theta^0} (1-\alpha(\vartheta))(f^{ne}(\vartheta) - b^{ne}(\vartheta))d\vartheta}{\int_{\Theta^0} (1-\alpha(\vartheta))d\vartheta}$, $v^e = v^{ne}$*, and* $q^e(\phi) = q^{ne}(\phi)$*, each* ϕ*. Then, there is* T^* *such that there is a SSE* ϕ^* *with* $\Theta^{0e}(\phi^*) = \Theta^0$ *if and only if* $T \in (0, T^*]$

Proof. See Appendix 3. ∎

The assumptions of this Lemma are much stronger than required. All we actually need is that, at the unique SSE ϕ^* associated with the economy where all the individuals invest in education, $(U^e(\theta) - U^{ne}(\theta)) > 0$, each θ. This is a very mild restriction indeed. It turn out to be satisfied under the assumptions of Lemma 2, but also under several other sets of restrictions on the fundamentals. We state it directly as Assumption 3. Lemma 2 can, then, be interpreted as showing that there are (open sets of) economies satisfying the assumption.

Assumption 3: Let ϕ^* be the (unique) SSE of the economy $\xi \in \Xi$ with

$\Theta^{0e} = \Theta^0$. There is an interval $(0, T^*]$ such that $G(\theta, \phi^*, T; \xi) \geq 0$ for each $\theta \in \Theta$ if and only if $T \in (0, T^*]$.

THEOREM 2. *Under Assumptions 1-3, for each* $\xi \in \Xi^*$, *an open, dense subset of* Ξ, *there is an interior SSE* $\phi(T) \in C^1$, *with associated nonempty set* $\Theta^{0e}(\phi(T), T; \xi) \neq \Theta^0$, *for each* $T \in (T_\xi, T^\xi)$, *an open subset of* $(0, \infty)$.

Proof. See Appendix 3. ∎

The somewhat intricate statement of the Theorem should not obscure the main point: modulo some arbitrarily small adjustment of production and education cost functions, and of vacancy costs, there is an interval (T_ξ, T^ξ) for which there are interior SSE. As already explained, this is the best result we can hope for, from the qualitative viewpoint.

Remark 2. The proof of the theorem exploits appropriate perturbations of the pair v. This is just to simplify the proofs. We conjecture that they could be replaced by perturbations of $f^k(\theta)$, for each k, and by local perturbations of the functions $q^k(\phi^k)$ (preserving, if so required, its invariance across markets).

Remark 3. To fix ideas, assume that, for some interval of values $T > T_\xi$, say $(T_\xi, T^\xi]$, there is a unique value θ_T such that an individual invest in education if and only if, say, $\theta \geq \theta_T$. Assume that, at $T' > T^\xi$, there are two values θ_T^1, θ_T^2 such that both $\theta_T^1, \theta_T^2 \in G_{T'}^{-1}(0)$. We conjecture that an argument similar to the one exploited in Appendix 3 could be used to establish existence of SSE at $T > T'$, given that, at each interior SSE, $G^{-1}(0)$ is a discrete set. Evidently, this would require local perturbations of $q^k(\phi^k)$.

Define the set

$$\Xi_T = \{\xi \in \Xi | \text{ given } T, \text{ there is an interior SSE } \phi(\xi)\}.$$

Next, we study some properties of Ξ_T, for arbitrarily given $T > 0$. The results may be of some autonomous interest, but the next Theorem is mainly motivated as a step to discuss the efficiency properties of SSE.

THEOREM 3. *For each* $T > 0$, *the set* Ξ_T *is non-empty. Moreover,* Ξ_T *contains an open, dense subset* Ξ_T^{reg} *such that, for each* $\xi^\circ \in \Xi_T^{reg}$, *at each interior SSE* $\phi(\xi^\circ)$,

i. $G(\theta_\ell, \phi(\xi^\circ); \xi^\circ) \neq 0$ *and* $G(\theta_h, \phi(\xi^\circ); \xi^\circ) \neq 0$,

ii. $D_\phi \Phi(\phi; \xi^\circ)$ *has full rank,*

iii. the number of interior SSE is finite, and there is an open neighborhood $V(\xi^\circ) \subset \Xi_T^{reg}$ *such that interior SSE are described by a finite collection of* C^1 *maps,* $(\phi_1(\xi), ..., \phi_N(\xi))$, *for some N.*

Proof. (iii) follows immediately from (i, ii) and the IFT. (i, ii) are established in Appendix. ∎

As standard, we call *regular* an interior SSE such that $D_\phi \Phi(\phi; \xi^\circ)$ has full rank (implicitly, this requires that (i) above holds, otherwise, $\Phi(\phi; \xi^\circ)$ may be non differentiable). If each interior SSE of an economy is regular, we call the economy *regular*.

5. EFFICIENCY PROPERTIES OF THE SSE ALLOCATIONS

A natural notion of constrained Pareto optimality (CPO) would require that the equilibrium allocation cannot be improved upon by changing the set of people getting educated and the "market tightness" variables ϕ. Unfortunately, such a notion is not useful in our context. Evidently, for an allocation to be CPO, it has to be CPO contingent on the specific selection of the set Θ^e. However, in the canonical one-sector, random matching model, SSE are typically constrained Pareto inefficient, unless the "market power" weight β happens to satisfy the Hosios (1990) condition, i.e., $\beta = |\eta_q|$. Given Θ^e, our model reduces to a pair of disjoint random matching economies and, therefore, a necessary condition for a SSE to be CPO is $\beta = |\eta_{q^k(\phi^k)}|$, for each k. It follows that SSE allocations typically are not CPO, as long as β is treated as an exogenous parameter. This obscures the nature of inefficiencies specifically related to the educational choices of the agents, if any. Therefore, we propose a different concept, the notion of *Weak Constrained Pareto Optimality (WCPO).* With our definition, the planner can choose any measurable subset Θ^e. The associated pair ϕ, however, is the corresponding SSE. Evidently, without investments in education, interior SSE are trivially WCPO, because they are globally unique, and, therefore, the constraint set of the planner reduces to a single point, the SSE itself. Hence, the notion of WCPO is extremely weak, and, consequently, WCPO allocations do not have a strong appeal from the normative viewpoint. This criterion, however, is useful, because it allows us to pinpoint sources of inefficiency just related to the two-sector structure of the economy, and to the private investments in education. Our notion of WCPO is somewhat related to the concept of CPO commonly used in the literature on general equilibrium with incomplete markets (see, Geanakoplos and Polemarchakis (1986)). In both cases the planner chooses the investment portfolios, taking into account the consequent adjustment of the endogenous equilibrium variables (prices there, ϕ here).

We restrict the analysis to steady states, and we assume that $r\prime = 0$, so that $r = \gamma$ (or, rather, we consider the limit case for r' converging to zero). This entails a loss of generality, but it allows us to sidestep issues related to dynamic optimality versus optimality of the steady states.

The planner's objective function, $P'(u, o, \Theta^{0e}; \xi)$, is the (discounted) expected total surplus, net of vacancy costs and of the direct costs of education of the cohort born in a given period t. His policy instruments are the choice of a measurable subset of Θ and of the pair (u, o). The planner faces three constraints:

1. $u^e = \frac{\gamma}{\gamma + \pi^e(\phi^e)}$;
2. $u^{ne} = \frac{\gamma}{\gamma + \pi^{ne}(\phi^{ne})}$;
3. $\Phi_{\Theta^e}(\phi; \xi) = 0$.

The last constraint may differ from the equilibrium condition $\Phi(\phi; \xi) = 0$, because, in $\Phi_{\Theta^e}(\phi; \xi)$, Θ^e is selected by the planner, while, in $\Phi(\phi; \xi)$, it is implicitly given by the additional condition $G(\theta, \phi; \xi) \geq 0$, for each $\theta \in \Theta^e$. Given the constraints (1-2), the policy instruments actually reduce to Θ^e and to the choice of the measure of job openings. Also, notice that we are implicitly imposing symmetry in the treatment of agents of the same type θ.

Define the function

$$
\begin{aligned}
T(\rho,\theta,\phi;\xi) &= \alpha(\theta)\frac{\rho\pi^{e}(\phi^{e}) f^{e}(\theta)+\gamma b^{e}(\theta)}{\gamma+\rho\pi^{e}(\phi^{e})}+\left(1-e^{\gamma T}\right)c(\theta) \\
&\quad +\left(1-e^{\gamma T}-\alpha(\theta)\right)\frac{\rho\pi^{ne}(\phi^{ne})f^{ne}(\theta)+\gamma b^{ne}(\theta)}{\gamma+\rho\pi^{ne}(\phi^{ne})}.
\end{aligned}
$$

Setting $\rho=\beta$, one obtains $T(\beta,\theta,\phi;\xi)=G(\theta,\phi;\xi)$. On the other hand, at $\rho=1$, $T(1,\theta,\phi;\xi)$ is the *social* gain (net of direct and opportunity costs) of the investment in education of agent θ, i.e., the relevant variable from the planner's viewpoint.

Integrating the steady state values of the variables, and replacing in the values of u^{k*} given by the constraints $1-2$, $P'(u,o,\Theta^{0e};\xi)$ can be rewritten as

$$
\begin{aligned}
P(\phi,\Theta^{0e};\xi) &= \left(e^{-\gamma T}\int_{\Theta^{0e}}T(1,\vartheta,\phi;\xi)\,d\vartheta+\frac{\int_{\Theta}\left[\pi^{ne}(\phi^{ne})f^{ne}(\vartheta)+\gamma b^{ne}(\vartheta)\right]d\vartheta}{\gamma+\pi^{ne}(\phi^{ne})}\right) \\
&\quad -e^{-\gamma T}\frac{\gamma v^{e}\phi^{e}\int_{\Theta^{0e}}\alpha(\vartheta)\,d\vartheta}{\gamma+\pi^{e}(\phi^{e})} \\
&\quad -\frac{\gamma v^{ne}\phi^{ne}\left[\int_{\Theta\setminus\Theta^{0e}}d\vartheta+e^{-\gamma T}\int_{\Theta^{0e}}(1-\alpha(\vartheta))\,d\vartheta\right]}{\gamma+\pi^{ne}(\phi^{ne})},
\end{aligned}
$$

where, for instance, $e^{-\gamma T}v^{e}\frac{\gamma\phi^{e}}{\gamma+\pi^{e}(\phi^{e})}\int_{\Theta^{0e}}\alpha(\vartheta)\,d\vartheta=v^{e}o^{e}\mu(\Theta^{e\alpha})$ is the total cost of job openings created (at time $(t+T)$) on market e. Hence, the last two terms describe the vacancy costs, given the sets of people getting/not getting an education. The first term in brackets is the expected output at the stationary allocation.

For completeness, we formally report the standard notion of CPO and the inefficiency result already mentioned.

Definition 2. A steady state pair (ϕ,Θ^{e}) is *Constrained Pareto Optimal* (CPO) if and only if it is a steady state solution to the optimization problem

$$\textit{choose }(\phi,\Theta^{e})\in\arg\max P(\phi,\Theta^{e};\xi).$$

Proposition 2. *There is an open, dense subset* $\Xi^{'}\subset\Xi$ *such that, for each* $\xi\in\Xi'$, *an interior SSE, if it exists, is not CPO.*

Proof. Assume that $|\eta_{q^{k}(\phi^{k})}|\neq\beta$, for each k. Then, given any $\Theta^{k\alpha}$, the result follows by a standard argument. It is straightforward to show that, generically, at a SSE, $|\eta_{q^{k}(\phi^{k})}|\neq\beta$, for each k. ∎

We obtain the notion of WCPO by introducing in the planner's optimization problem the additional constraint (3) discussed above.

Definition 3. A steady state pair (ϕ,Θ^{e}) is *Weakly Constrained Pareto Optimal* (WCPO) if and only if it is a steady state solution to the optimization problem

$$\textit{choose }(\phi,\Theta^{e})\in\arg\max P(\phi,\Theta^{e};\xi)\ \textit{subject to}\ \Phi_{\Theta^{e}}(\phi;\xi)=0.$$

Theorem 4. *Under the maintained assumptions, there is an open, dense subset of economies,* $\Xi"\subset\Xi$, *such that, for each* $\xi\in\Xi"$, *every regular interior SSE allocation, if it exists, is not WCPO.*

Proof. See Appendix 4. ∎

Remark 4. Throughout the paper, β is considered as an exogenous parameters. As we will see later on, the value of $(\beta+\eta_{q^k(\phi^k)})$ plays a role in determining the lack of WCPO of SSE and, most important, the nature of the inefficiency. However, it is neither necessary, nor sufficient to restore WCPO. In fact, this conditions plays no role in the proof of Theorem 4.

Remark 5. We are completely agnostic about the (far from trivial) problem of the existence of WCPO allocations, that is not really germane to the issue under consideration.

Remark 6. The proof of Theorem 4 holds for each regular interior SSE. We have formally established existence of this sort of equilibria for a (possibly) small subset of economies. However, this last theorem does not rest in any substantive sense on the proof of Theorem 2. Its result holds for *all* the regular interior SSE. Moreover, its proof rests heavily on differentiability. It is worthwhile to stress that this property is never at issue here: In the "planner's problem" what matters are the derivatives $\frac{\partial \phi^k}{\partial \theta_m}$, $k = e, ne$, obtained by the IFT applied to the constraint $\Phi_{\Theta^e}(\phi;\xi) = 0$, at the values $\theta_m \in G^{-1}(0)$. While it is possible that $rank\ D_\phi \Phi(\phi;\xi) < 2$, $rank D_\phi \Phi_{\Theta^e}(\phi;\xi) = 2$, always. Therefore, $\frac{\partial \phi^k}{\partial \theta_m}$, $k = e, ne$, are always well-defined, now.

In the literature, three different possible sources of (constrained) inefficiency have been identified. First, as pointed out in Hosios (1990), when $\beta \neq |\eta_{q^k(\phi^k)}|$, SSE are inefficient because agents do not internalize the congestion externality. In particular, in the basic, one-sector model, when $\beta > |\eta_{q^k(\phi^k)}|$, the SSE ϕ is below its optimal value (hence, the rate of unemployment is above its optimal level).

Secondly, with investments in human capital, there may be an "hold up" effect, stressed by Acemoglu (1996): Educated workers do not receive the full return on their investment, because of the noncompetitive wage determination mechanism and of the irreversible nature of their investment. In his model, this induces underinvestment in education.

A third possible cause of inefficiency may be related to the "composition effect". Assume that both $f^k(\theta)$ are strictly increasing in θ and that there is a unique $\theta^* \in G^{-1}(0)$. Moreover, assume that only agents with $\theta \geq \theta^*$ invest in education. Evidently, agent θ^* is, simultaneously, the most productive uneducated worker and the least productive educated one. This is a potential source of inefficiency due to overinvestment in education, as pointed out in Charlot and Decreuse (2005). In particular, to move up the threshold value θ^* increases the equilibrium pair ϕ, and this can be Pareto improving.

With our notion of WCPO, congestion externalities market by market are neutralized. The other two kinds of sources of inefficiency are potentially active. Moreover, the sign of $(\beta + \eta_{q^k(\phi^k)})$ may affect the type of inefficiency (over versus undereducation).

Let's make precise our notions of over and undereducation. As in the proof of Theorem 4, let's restrict the planner to choose sets Θ^e given by the union of a finite collection of intervals $[\theta_m, \theta_{m+1}]$. Replace into the planner's objective function the pair ϕ, implicitly given by the constraint $\Phi_{\Theta^e}(\phi;\xi) = 0$, a C^1 function of the vector $[\theta_1, ..., \theta_m]$, $\phi(\theta_1, ..., \theta_m)$. The modified planner's optimization problem is, then,

$$\max_{[\theta_1,...,\theta_m]} P^*(\theta_1, ..., \theta_m; \xi) \equiv P(\phi^e(\theta), \phi^{ne}(\theta); \xi), \tag{10}$$

Also, define $\chi(\theta) = 1$, if $\theta_m \in [\theta_m, \theta_{m+1}] \subset \Theta^{0e}$, $\chi(\theta) = 2$, if $\theta_m \in [\theta_{m-1}, \theta_m] \subset \Theta^{0e}$.

Definition 4. A SSE of the economy $\xi \in \Xi$ exhibits (local) *undereducation* at $\theta_m \in G^{-1}(\theta_m)$ if and only if $(-1)^{\chi(\theta_m)} \frac{\partial P^*}{\partial \theta_m} > 0$. It exhibits (local) *overeducation* at $\theta_m \in G^{-1}(\theta_m)$ if and only if $(-1)^{\chi(\theta_m)} \frac{\partial P^*}{\partial \theta_m} < 0$.

This formulation will become handy in the sequel. We have overeducation if (locally) we increase the total net surplus by shrinking the set of agents investing in education. If θ_m is the lower bound of an interval $[\theta_m, \theta_{m+1}] \subset \Theta^{0e}$, this means that $\frac{\partial P^*}{\partial \theta}|_{\theta_m} > 0$, if θ_m is an upper bound, it means $\frac{\partial P^*}{\partial \theta_m} < 0$, i.e., it means $(-1)^{\chi(\theta_m)} \frac{\partial P^*}{\partial \theta_m} < 0$.

The (necessary) first order conditions of the modified planner's optimization problem (10) include

$$\frac{\partial P^*}{\partial \theta_m} = \frac{\partial P}{\partial \theta_m} + \left(\frac{\partial P}{\partial \phi^e} \frac{\partial \phi^e}{\partial \theta_m} + \frac{\partial P}{\partial \phi^{ne}} \frac{\partial \phi^{ne}}{\partial \theta_m} \right) = 0, \text{ for each } \theta_m \in G^{-1}(0) \cap (\theta_\ell, \theta_h).$$

Thus, $\frac{\partial P^*}{\partial \theta_m}$ is the sum of two terms, capturing the direct and indirect effects of changes in θ_m on the objective function. By direct computation,

$$(-1)^{\chi(\theta m)} e^{\gamma T} \frac{\partial P}{\partial \theta_m} = T(1, \theta_m, \phi; \xi) - \left(\frac{\alpha(\theta_m) \gamma v^e \phi^e}{\gamma + \pi^e(\phi^e)} + \frac{\left(1 - e^{\gamma T} - \alpha(\theta_m)\right) \gamma v^{ne} \phi^{ne}}{\gamma + \pi^{ne}(\phi^{ne})} \right),$$

where $T(1, \theta_m, \phi; \xi)$ is the change in total expected (discounted) output due to the investment in education of agent θ_m (net of direct and opportunity costs). This term is related to the "hold up" problem stressed in Acemoglu (1996), because, when $\beta = 1$, $T(1, \theta_m, \phi; \xi) = T(\beta, \theta_m, \phi; \xi) = 0$. In general, assume that $b^k(\theta) = 0$. Then, it is easy to see that $T(1, \theta_m, \phi; \xi)$ is positive if $\pi^e(\phi^e)$ is not "too large" compared to $\pi^{ne}(\phi^{ne})$[7]. The second term is the difference in discounted expected vacancy costs in the two markets. If it is sufficiently small, and $(\pi^e(\phi^e) - \pi^{ne}(\phi^{ne}))$ not too large, $(-1)^{\chi(\theta)} \frac{\partial P^{dir}}{\partial \theta_m} > 0$, so that the direct effect induces undereducation. On the other hand, if $(\pi^e(\phi^e) - \pi^{ne}(\phi^{ne}))$ is positive and sufficiently large, we may have $(-1)^{\chi(\theta)} \frac{\partial P^{dir}}{\partial \theta_m} < 0$. Bear in mind that the direct effect does not depend in any way upon the value of β, and it can be different from zero even if the Hosios condition holds, for each k.

The second, indirect, component is related to the effect of changes in θ_m on the equilibrium values of the market tightness variables, ϕ. By direct computation, given that, at a SSE, $\Phi^k(\phi; \xi) = 0$, and rearranging terms, we obtain

$$(-1)^{\chi(\theta_m)} \frac{\partial P}{\partial \phi^k} \frac{\partial \phi^k}{\partial \theta_m} = \frac{\gamma v^k \mu\left(\Theta^{k\alpha}\right) \left(\beta + \eta_{q^k(\phi^k)}\right)}{(1-\beta)\left(\gamma + \pi^k\left(\phi^k\right)\right)} \left((-1)^{\chi(\theta_m)} \frac{\partial \phi^k}{\partial \theta_m} \right), \text{ each } k.$$

[7] At each ρ such that $T(\rho, \theta, \phi; \xi) = 0$,

$$\frac{\partial T}{\partial \rho} = \frac{\gamma \left[\frac{\alpha(\theta_m)\rho\pi^e(\phi^e) f^e(\theta_m)}{\gamma + \rho\pi^e(\phi^e)} + \left(1 - e^{\gamma T} - \alpha(\theta_m)\right) \times \frac{\pi^{ne}(\phi^{ne})\rho f^{ne}(\theta_m)}{\gamma + \rho\pi^{ne}(\phi^{ne})} \frac{\gamma + \rho\pi^e(\phi^e)}{\gamma + \rho\pi^{ne}(\phi^{ne})} \right]}{\rho(\gamma + \rho\pi^e(\phi^e))} > 0$$

for $\pi^e(\phi^e)$ sufficiently close to (or smaller than) $\pi^{ne}(\phi^{ne})$. Given that $T(\rho = \beta, \theta, \phi; \xi) = 0$, this implies $T(1, \theta, \phi; \xi) > 0$.

This term is nil if and only if either Hosios condition holds or $\frac{\partial \phi^k}{\partial \theta_m} = 0$. The Hosios condition comes back into play because of the change of the pair ϕ induced by the change in the value of θ, even if our notion of efficiency is constructed to neutralize the canonical (i.e., given $\Theta^{k\alpha}$, each k) Hosios effect. By the IFT, and direct computation,

$$\begin{aligned}
\frac{\partial \phi}{\partial \theta_m} &= -\left[\frac{\partial \Phi_{\Theta^e}}{\partial \phi}\right]^{-1}\left[\frac{\partial \Phi_{\Theta^e}}{\partial \theta_m}\right] \\
&= (-1)^{\chi(\theta_m)} \frac{(1-\beta)}{e^{\gamma T}} \left[\begin{array}{c} \frac{\alpha(\theta_m)}{\gamma+\pi^e(\phi^e)} \frac{(f^e(\theta_m)-b^e(\theta_m))-F^e\left(\Omega^{0e}\right)}{\mu(L^e)} \left(-\frac{\partial \Phi^e_{\Theta^e}}{\partial \phi^e}\right)^{-1} \\ \frac{1-e^{\gamma T}-\alpha(\theta_m)}{\gamma+\pi^{ne}(\phi^{ne})} \frac{(f^{ne}(\theta_m)-b^{ne}(\theta_m))-F^{ne}\left(\Omega^{0e}\right)}{\mu(L^{ne})} \left(-\frac{\partial \Phi^{ne}_{\Theta^e e}}{\partial \phi^{ne}}\right)^{-1} \end{array} \right],
\end{aligned}$$

where $F^k\left(\Omega^{0e}\right) = \frac{\int_{\Omega^{e\alpha}} (f^e(\vartheta)-b^e(\vartheta))d\vartheta}{\mu(L^k)}$. It is easy to check that $\frac{\partial \Phi^k_{\Theta^e}}{\partial \phi^k} < 0$, each k. Given that $\left(1-e^{\gamma T}-\alpha\left(\theta_m\right)\right) < 0$,

$$sign \left[\begin{array}{c} \frac{\partial \phi^e}{\partial \theta_m} \\ \frac{\partial \phi^{ne}}{\partial \theta_m} \end{array} \right] = sign\,(-1)^{\chi(\theta_m)} \left[\begin{array}{c} \left(\left(f^e\left(\theta_m\right)-b^e\left(\theta_m\right)\right)-F^e\left(\Omega^{0e}\right)\right) \\ -\left(\left(f^{ne}\left(\theta_m\right)-b^{ne}\left(\theta_m\right)\right)-F^{ne}\left(\Omega^{0e}\right)\right) \end{array} \right].$$

Generally speaking, it is very hard to discriminate between overeducation and undereducation. This is also because, when there are several $\theta_m \in G^{-1}(0)$, in general, the SSE is characterized by overeducation (i.e., $\frac{\partial P^*}{\partial \theta_m} < 0$) at some θ_m, by undereducation at some other $\theta_{m'}$. However, at least one important point is established: In a two-sector economy, the Hosios condition is neither necessary, nor sufficient, to guarantee that SSE allocations are constrained Pareto efficient. This is a sharp difference with respect to the results one obtains in the basic, one-sector, random matching model.

6. TWO POLAR CASES: ABILITY AND EDUCATION AS COMPLEMENTS AND SUBSTITUTES

To conclude, we focus the analysis on two polar cases where, at each SSE, there is a unique $\theta^* \in G^{-1}(0)$. We introduce an additional, simplifying, assumption,

Assumption 4: $b^k(\theta) = c(\theta) = 0$, each k and θ. Moreover, $\nu^e = \nu^{ne}$ and $q^e(\phi) = q^{ne}(\phi)$.

We start providing some restrictions on the fundamentals of the economy which give a (partial) characterization of complementarity *vs.* substitutability.

Let's define $\eta_\alpha(\theta) \equiv \frac{\partial \alpha(\theta)}{\partial \theta}\frac{\theta}{\alpha(\theta)}$, $\eta_{f^e}(\theta) \equiv \frac{\partial f^e(\theta)}{\partial \theta}\frac{\theta}{f^e(\theta)}$ and $\eta_{f^{ne}}(\theta) \equiv \frac{\partial f^{ne}(\theta)}{\partial \theta}\frac{\theta}{f^{ne}(\theta)}$.

LEMMA 3. *Under the maintained assumptions,*
a. complementarity between ability and education*: if* $\eta_\alpha(\theta) \geq 0$ *and* $\eta_{f^e}(\theta) > \eta_{f^{ne}}(\theta)$*, each* θ*, at each SSE* ϕ^**, there is a unique* $\theta \in G^{-1}(0)$ *and* $\Theta^{0e}(\phi^*) = [\theta(\phi^*), \theta_h]$;
b. substitutability between ability and education*: if* $\eta_{f^e}(\theta) < \eta_{f^{ne}}(\theta)$*, and* $\eta_\alpha(\theta)$ *is sufficiently small, each* θ*, at each SSE* ϕ^**, there is a unique* $\theta \in G^{-1}(0)$ *and* $\Theta^{0e}(\phi^*) = [\theta_\ell, \theta(\phi^*)]$.

Proof. Appendix 5. ∎

Remark 7. In the case of complementarity, only the high θ people invest in education. In the one of substitutability, only the low θ. A priori, both cases are plausible. Obviously, what matters are the comparative advantages. If $\frac{\eta_{f^{ne}}(\theta)}{\eta_{f^e}(\theta)} > 1$, each θ, the comparative advantage in the high skill job is decreasing in θ.

6.1. Constrained inefficiency

Let's first consider the direct effect $\frac{\partial P}{\partial \theta^*}$, computed at the unique $\theta^* \in G^{-1}(0)$. Using the simplifying assumptions, the direct effect of a change in θ^* on total surplus is

$$\begin{aligned} e^{\gamma T}(-1)^{\chi(\theta m)}\frac{\partial P}{\partial \theta^*} &= \left(\frac{\alpha(\theta^*)\pi(\phi^e) f^e(\theta^*)}{\gamma+\pi(\phi^e)} + \frac{\left(1-e^{\gamma T}-\alpha(\theta^*)\right)\pi(\phi^{ne}) f^{ne}(\theta^*)}{\gamma+\pi(\phi^{ne})}\right) \\ &\quad -\left(\frac{\alpha(\theta^*)\gamma v\phi^e}{\gamma+\pi(\phi^e)} + \left(1-e^{\gamma T}-\alpha(\theta^*)\right)\frac{\gamma v\phi^{ne}}{\gamma+\pi(\phi^{ne})}\right). \end{aligned}$$

In the case of complementarity, it is always $\pi(\phi^e) > \pi(\phi^{ne})$. The first term in brackets $(T(1,\theta^*,\phi;\xi)$, using the notation introduced above) is negative when $\pi(\phi^e) > \pi(\phi^{ne})$, because, at a SSE, $T(\rho,\theta^*,\phi;\xi) = 0$ if and only if $\rho = \beta$, and $\frac{\partial T}{\partial \rho} < 0$ at $\rho > \beta$[8]. Consider now the second term in brackets. Fix ϕ^{ne}. Under the maintained assumptions, $\frac{\phi}{\gamma+\pi(\phi)}$ is an increasing function of ϕ, unbounded above. Hence, for $\frac{\pi(\phi^e)}{\pi(\phi^{ne})}$ (i.e., $\frac{\phi^e}{\phi^{ne}}$) sufficiently large, this term is positive and, therefore, $(-1)^{\chi(\theta m)}\frac{\partial P}{\partial \theta^*} < 0$. Given the equilibrium conditions, and $\eta_{f^{ne}}$, a sufficient condition to obtain an arbitrarily large ratio $\frac{\phi^e}{\phi^{ne}}$ is to have η_{f^e} sufficiently large. Consider now the indirect effect. Under the maintained assumptions, $\frac{\partial \phi^k}{\partial \theta^*} > 0$, each k. Therefore, if $(\beta + \eta_{q^k(\phi^k)}) > 0$, each k,

$$(-1)^{\chi(\theta^*)}\frac{\partial P}{\partial \phi^k}\frac{\partial \phi^k}{\partial \theta^*} = (-1)^{\chi(\theta^*)}\sum_k \frac{\gamma v\mu\left(\Theta^{k\alpha}\right)\left(\beta+\eta_{q(\phi^k)}\right)}{(1-\beta)\left(\gamma+\pi\left(\phi^k\right)\right)}\frac{\partial \phi^k}{\partial \theta^*} < 0,$$

because $\chi(\theta^*) = 1$. Hence, the inefficiency of the SSE is due to overeducation.

The case of substitutability can be treated in similar way. We have established the following:

Proposition 3. *Assume that, at the SSE, $\beta \geq |\eta_{q(\phi^k)}|$, for each k. Then, given assumptions (1-4):*
a. complementarity: if η_{f^e} is sufficiently large, the SSE is characterized by overeducation;
b. substitutability: if $\eta_{f^{ne}}$ is sufficiently large, the SSE is characterized by undereducation.

Consider again the case of complementarity. A sufficiently large value of $(\pi(\phi^e) - \pi(\phi^{ne}))$ (induced by high η_{f^e}) is required just because $T > 0$ and $\alpha(\theta^*) < 1$. It

[8] This follows from two observations. First, $T(\rho,\theta_m,\phi;\xi) = 0$ if and only if $\frac{(\gamma+\pi^e(\phi^e))(\gamma+\pi^{ne}(\phi^{ne}))}{\rho}T(\rho,\theta_m,\phi;\xi) = 0$. This last equation is linear in ρ, and it has a unique solution, $\beta = \rho$. Given that, at $\rho = \beta$, $\frac{\partial T(\rho,\theta_m,\phi;\xi)}{\partial \rho} < 0$, if $(\pi^e(\phi^e) - \pi^{ne}(\phi^{ne})) < 0$, the result follows.

is easy to check that, for $T = 0$, $\alpha(\theta^*) = 1$, and (evidently) $c(\theta^*) > 0$, the direct effect of an increase in θ^* is always Pareto improving. When $\beta \geq |\eta_{q(\phi^k)}|$, each k, in each sector unemployment is above its constrained Pareto optimal level. The indirect effect of an increase in θ^* is a reduction in unemployment in both sectors, and, therefore, a Pareto improvement. When $\beta < |\eta_{q(\phi^k)}|$, unemployment is below its CPO level. Hence, in this case, an increase in θ^* may have a positive direct effect on welfare, but it has always a negative indirect one, so that the sign of the total effect is undefined.

According to the results reported in Petrongolo and Pissarides (2001, p. 393) the range of the most plausible values of $\eta_{q(\phi^k)}$ is $(-0.5, -0.3)$. The value of β has been estimated for several countries. Most of the results suggest that its value is fairly small (see Yashiv (2003, 2006), Cahuc, Postel-Vinay, and Robin (2006) and other references quoted therein[9]). It follows that the case considered in Proposition 3 may not be, empirically, the most relevant one.

6.2. Comparative statics of regular equilibria

By the IFT, $D_\xi \phi = -D_\phi \Phi(\theta, \phi; \xi)^{-1} D_\xi \Phi(\theta, \phi; \xi)$. Hence, we restrict the analysis to the (generic) subset of regular economies.

It is very convenient to replace the actual SSE map $\Phi(.)$ with

$$\begin{aligned}
\Phi'(\phi;\xi) &\equiv \begin{bmatrix} \frac{r+\beta\pi^e(\phi^e)}{1-\beta}\Phi^e(\phi;\xi) \\ \frac{r+\beta\pi^{ne}(\phi^{ne})}{1-\beta}\Phi^{ne}(\phi;\xi) \end{bmatrix} \equiv \begin{bmatrix} F^e(\theta^*(\phi)) - A^e(\phi^e) \\ F^{ne}(\theta^*(\phi)) - A^{ne}(\phi^{ne}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\int_{L^e(\phi)}(f^e(\vartheta)-b^e(\vartheta))d\vartheta}{\mu(L^e(\phi))} - \frac{rv^e+\beta v^e\pi^e(\phi^e)}{(1-\beta)q^e(\phi^e)} \\ \frac{\int_{L^{ne}(\phi)}(f^{ne}(\vartheta)-b^{ne}(\vartheta))d\vartheta}{\mu(L^{ne}(\phi))} - \frac{rv^{ne}+\beta v^{ne}\pi^{ne}(\phi^{ne})}{(1-\beta)q^{ne}(\phi^{ne})} \end{bmatrix}.
\end{aligned}$$

Using the chain rule (and $\Phi(\phi;\xi) = 0$),

$$\frac{\partial \phi}{\partial \xi} = -D_\phi \Phi(\theta, \phi; \xi)^{-1} D_\xi \Phi(\theta, \phi; \xi) = -D_\phi \Phi'(\theta, \phi; \xi)^{-1} D_\xi \Phi'(\theta, \phi; \xi).$$

The main advantage of this transformation is that, for each k, $F^k(\theta^*(\phi))$ depends upon ϕ only because of the effects of its changes on the value of θ^*, while, for each k, $A^k(\phi^k)$ only depends upon ϕ^k.

By direct computation,

$$D_\phi \Phi'(\phi;\xi)^{-1} = \frac{1}{\det D_\phi \Phi'} \begin{bmatrix} \frac{\partial F^{ne}}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^{ne}} - \frac{\partial A^{ne}}{\partial \phi^{ne}} & -\frac{\partial F^e}{\partial \theta^*}\frac{\partial \theta}{\partial \phi^{ne}} \\ -\frac{\partial F^{ne}}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^e} & \frac{\partial F^e}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^e} - \frac{\partial A^e}{\partial \phi^e} \end{bmatrix}.$$

Intuitively, the comparative static properties of the economy rest heavily on the sign of $\det D_\phi \Phi'(.)$, where

$$\det D_\phi \Phi'(.) = \frac{\partial A^e}{\partial \phi^e}\frac{\partial A^{ne}}{\partial \phi^{ne}} - \frac{\partial A^{ne}}{\partial \phi^{ne}}\frac{\partial F^e}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^e} - \frac{\partial A^e}{\partial \phi^e}\frac{\partial F^{ne}}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^{ne}}.$$

The first term is always positive. With complementarity, the second is negative and the third is positive. Under substitutability, the third term is negative, while the second is positive.

[9] Notice, however, that Cahuc, Postel-Vinay, and Robin (2006) reports values of β^{ne} around 0.1, but substantially larger values for β^e. Also, Flinn and Mabli (2008) reports relatively high values of β.

We will only consider the case when the total effect (i.e., inclusive of its impact on the average ability of the pool of workers) of a change in ϕ^k on the (ex-ante) actualized profits in sector k is negative, i.e., $\left(\frac{\partial F^k}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^k} - \frac{\partial A^k}{\partial \phi^k}\right) < 0$. With complementarity, $\left(\frac{\partial F^{ne}}{\partial \theta}\frac{\partial \theta^*}{\partial \phi^{ne}} - \frac{\partial A^{ne}}{\partial \phi^{ne}}\right) < 0$ implies $\det D_\phi \Phi'(.) > 0$. With substitutability, $\left(\frac{\partial F^e}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^e} - \frac{\partial A^e}{\partial \phi^e}\right) < 0$ implies $\det D_\phi \Phi'(.) > 0$.

Notice that the (different) restrictions we impose (and which deliver us a positive determinant) are, in both cases, coherent with the other maintained assumptions: $\left(\frac{\partial F^{ne}}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^{ne}} - \frac{\partial A^{ne}}{\partial \phi^{ne}}\right)$ is negative if $\eta_{F^{ne}}$ is sufficiently small. Given that $\mu(L^{ne})$ is bounded away from zero, one can check that $\eta_{F^{ne}}$ is an increasing function of $\eta_{f^{ne}}$, which is required (in the case of complementarity) to be itself small. Substitutability is characterized, *inter alia*, by $\eta_{f^e}(\theta) < \eta_{f^{ne}}(\theta)$. If $\mu(L^e)$ is bounded away from zero, η_{F^e} is an increasing function of η_{f^e}, which is assumed to be (comparatively) small.

We define the shocks to technologies, direct costs of education, probability of graduation and matching function as before (see Proposition 1), in terms of a multiplicative change. Shocks to vacancy costs are defined in the obvious way. For the functions describing direct costs of education and home productions, we focus on the case of θ-invariant, additive shocks.

PROPOSITION 4. *Under the maintained assumptions, if* $\left(\frac{\partial F^{ne}}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^{ne}} - \frac{\partial A^{ne}}{\partial \phi^{ne}}\right) < 0$, *and* $\frac{\partial F^{ne}}{\partial \theta^*} > 0$, *each* θ, *in the case of complementarity,*

$$\begin{bmatrix} \frac{\partial \phi^k}{\partial \xi}\diagdown d\xi & df^e & df^{ne} & dc & d\alpha & db^e & db^{ne} & dv^e & dv^{ne} & dq^e & dq^{ne} \\ \frac{\partial \phi^e}{\partial \xi} & ? & + & + & - & - & ? & - & - & - & ? \\ \frac{\partial \phi^{ne}}{\partial \xi} & - & + & + & - & ? & ? & + & - & ? & ? \end{bmatrix}.$$

Proof. Appendix 5. ∎

The additional restriction $\frac{\partial F^{ne}}{\partial \theta^*} > 0$ is certainly satisfied (in the case of complementarity) if $\alpha(\theta)$ is sufficiently close to 1, each θ.

The results above can be easily interpreted in terms of the Charlotte-Decreuse's composition effect. Changes in the exogenous parameters making, *coeteris paribus*, the market for uneducated workers more attractive (i.e., $df^{ne} > 0$, $dc > 0, d\alpha < 0, dv^{ne} < 0$) always increase both ϕ^e and ϕ^{ne}. This is because they attract (comparatively) higher ability individuals to this market, improving the expected product in both sectors. On the other hand, consider for instance, a positive shock to the technology in the educated labor market. The highest ability workers (among the ones who did not previously invest in education) are now attracted to this market. This immediately reduces ϕ^{ne}. Moreover, their productivity is lower than the one of the other educated workers, and this reduces the expected product in the market for educated workers (hence, the equilibrium level of ϕ^e). Therefore, the positive effect on ϕ^e of the shock is partly (or completely) counterbalanced by the (negative) composition effect.

The case of substitutability can be discussed in a similar way.

PROPOSITION 5. *Under the maintained assumptions, under substitutability, if*

$\left(\frac{\partial F^e}{\partial \theta^*}\frac{\partial \theta^*}{\partial \phi^e} - \frac{\partial A^e}{\partial \phi^e}\right) < 0,$

$$\begin{bmatrix} \frac{\partial \phi^k}{\partial \xi} \diagdown d\xi & df^e & df^{ne} & dc & d\alpha & db^e & db^{ne} & dv^e & dv^{ne} & dq^e & dq^{ne} \\ \frac{\partial \phi^e}{\partial \xi} & + & - & - & + & ? & ? & - & + & ? & - \\ \frac{\partial \phi^{ne}}{\partial \xi} & + & ? & - & + & ? & - & - & - & ? & - \end{bmatrix}.$$

Proof. Appendix 5. ∎

7. CONCLUSION

We have provided a fairly exhaustive, theoretical analysis of a two-sector economy where heterogeneous agents invest optimally in education, providing a generalization of the canonical Roy (1951) model to random matching environments.

From a generic viewpoint, the model is well-defined (i.e., there is a SSE under some restrictions on the - exogenous - length of the education process). Interior SSE, when they exist, have well-defined properties in terms of (lack of) efficiency. Given the technique of proof adopted, these properties are robust to many possible extensions of the model. More stylized (but still fairly general) versions of the model allow for (reasonably) sharp comparative static properties of SSE, and for a partial characterization of inefficiency in terms of overeducation or undereducation. The nexus between comparative statics properties and the nature of inefficiency makes the model potentially testable.

A key feature of the model is the role of the composition effect. The parametric set-up of Charlot and Decreuse (2005) allows them to obtain sharper conclusions, that do not necessarily hold for our more general class of economies. However, their essential message is confirmed. In a model with frictions, education allows agents to self-select themselves in one of the labor markets[10]. Generically, this has relevant consequences, which are ruled out by assumption in economies where investments in education translate into an increase in the number of efficiency units of the labor endowments.

An essential ingredient of our model is the assumption that matching is at random. With directed search, the inefficiency results may not survive. The extension of our analysis to economies with directed search is still an open issue. As it is, this paper could also be seen as a contribution to the literature on hybrid models of matching, i.e., models characterized by partially directed search.

8. APPENDICES

8.1. Appendix 1: Transversality

Several of the proofs are applications of the transversality theorem. We reproduce here the key mathematical results that we are going to exploit.

Let N and M be smooth manifolds of dimension n and m, respectively, and with $0 \in M \subset \mathbb{R}^m$. A smooth function $\Psi : N \to M$ is transverse to 0, $\Psi \pitchfork 0$ if, whenever $x \in N$ satisfies $\Psi(x) = 0$, $rank D_x \Psi = m$. Clearly, if $n < m$, $\Psi \pitchfork 0$ means that there is no solution to $\Psi(x) = 0$, i.e., that $\Psi^{-1}(0) = \emptyset$. If $m = n$, it means that $\Psi^{-1}(0)$ is a discrete set.

[10] An extension of the model to the N-sectors case would be far from trivial.

Let's now assume that $n = k + m$ and let $N = N_k \times N_m$, with $y \in N_m$ and $x \in N_k$. N_k and N_m are smooth manifolds of dimension k and m, respectively, where only N_k may have a boundary.

Transversality Theorem (TT). *Let $\Psi(x, y) \pitchfork 0 \in M \subset \mathbb{R}^m$. Then, except for a null subset $N_m^* \subset N_m$, if $\Psi^{-1}(0) \subset N_k \times N_m^*$, then $\Psi_y \pitchfork 0$ for each $x \in N_k$. Moreover, for each compact subset $N_k^C \subset N_k$, if $\Psi^{-1}(0) \subset N_k^C \times N_m$, the associated set N_m^* is an open, dense subset of N_m.*

Obviously, Ξ is not a finite dimensional space. However, we will always use local perturbations which are polynomial, i.e., finite dimensional. To go from our results to results referred to Ξ is a purely technical, and straightforward, matter.

We will exploit the transversality Theorem in several different context. Therefore, we outline here the general procedure we are going to adopt. To apply TT, we exploit (arbitrarily small) perturbations of the vector ν and of the functions (f, b, c). We start with a given function, say $f^k(\theta)$, and introduce a polynomial perturbation, setting

$$f^k(\theta; d) = f^k(\theta) + \sum_{v=0}^{V} d_v^k \theta^v,$$

where $d \in D \subset \mathbb{R}^V$, some small open neighborhood of 0. Using the theorem, and some additional properties, we show that a required property holds for all the vectors $d \in D^*$, some open, dense subset of D. This is what we exactly mean saying that a property holds "generically in $f^k(\theta)$" or "modulo a perturbation of $f^k(\theta)$". To use polynomial (hence, finite dimensional) perturbations is convenient, and it does not imply any essential loss of generality with respect to openness and density of the set of functions we restrict ourselves to.

Finally, we can also perturb in different directions the same function: Pick, for instance, θ_1, θ_2, with $\theta_1 \neq \theta_2$. Choose two open neighborhoods of radius $\varepsilon, V_\varepsilon(\theta_1)$ and $V_\varepsilon(\theta_2)$, such that $clV_{2\varepsilon}(\theta_1) \cap clV_{2\varepsilon}(\theta_2) \neq \emptyset$. Choose two "bump" functions (with nonnegative values) $\psi_1(\theta)$ and $\psi_2(\theta)$, taking the value 1 on the set $V_\varepsilon(\theta_1)$ $(V_\varepsilon(\theta_2))$ and the value 0 at $\theta \notin clV_{2\varepsilon}(\theta_1)$ $(clV_{2\varepsilon}(\theta_2))$. Functions with these properties exist (see, Hirsch (1976, p. 41-43)). Define the perturbed function $f^k(\theta; d) = f^k(\theta) + \psi_1(\theta) d_1 + \psi_2(\theta) d_2$. Evidently, on, say, $V_\varepsilon(\theta_1)$, $\frac{\partial f^k(\theta;d)}{\partial d_1} = 1$ and $\frac{\partial f^k(\theta;d)}{\partial d_1} = 0$ at $\theta \notin clV_{2\varepsilon}(\theta_1)$. In a similar way, and using polynomials in d_1 and d_2, we can arbitrarily (and independently) perturb the derivatives of any order of the functions.

8.2. Appendix 2: Comparative statics of the aggregate demand for education

Proof of Lemma 1. We start showing that, given T, for an open, dense set of economies $\Xi' \subset \Xi$, at each $\phi \in F_\xi$, a compact subset of $\mathbb{R}^2_{++}$, $G^{-1}(0)$ contains a finite number of isolated points.

First, by contradiction, we show that Ξ' is a dense set. Otherwise, there is some open neighborhood $V(\xi^\circ) \subset \Xi \setminus \Xi'$. Without loss of generality, assume that F_ξ is a smooth, compact manifold contained in $\mathbb{R}^2_{++}$. Define the system of equations

$$\Psi(\theta, \phi; \xi) \equiv \begin{pmatrix} G(\theta, \phi; \xi) \\ \frac{\partial G}{\partial \theta} \\ \frac{\partial^2 G}{\partial \theta^2} \\ \frac{\partial^3 G}{\partial \theta^3} \end{pmatrix} = 0,$$

$\Psi : [\theta_\ell, \theta_h] \times \digamma_\xi \times V(\xi^\circ) \to \mathbb{R}^4$. Define $c(\theta; d) = c(\theta) + \sum_{j=0}^{3} d_j \theta^j$, for $d = (d_0, \ldots, d_3)$ in some open set $D \subset \mathbb{R}^4$, such that $0 \in D$, and $c(\theta; d) > 0$, for each $\theta \in [\theta_\ell, \theta_h]$. Pick D sufficiently small, so that the economies obtained with a $d-$perturbation of ξ° lie in $V(\xi^\circ) \subset \Xi \setminus \Xi'$.

It is easy to check that $D_d \Psi(\theta, \phi; \xi)$ is a full rank matrix. Hence, $\Psi \pitchfork 0$, and by the transversality theorem, there is a full measure subset $D' \subset D$ such that $\Psi_d \pitchfork 0$, for each $d \in D'$. Given that Ψ_d maps a subset of $\mathbb{R}^3$ into $\mathbb{R}^4$, $\Psi_d^{-1}(0) = \emptyset$. Therefore, at each $\theta \in G_d^{-1}(0)$, either $\frac{\partial G_d}{\partial \theta} \neq 0$, or, if $\frac{\partial G_d}{\partial \theta} = 0$, either $\frac{\partial^2 G_d}{\partial \theta^2} \neq 0$ or $\frac{\partial^3 G_d}{\partial \theta^3} \neq 0$ or both. It follows that $G_d^{-1}(0)$ does not contain any degenerate local extrema and, therefore, that it is a discrete set. Hence, by compactness, $G_d^{-1}(0) \cap [\theta_\ell, \theta_h]$ is a finite set. The contradiction establishes that Ξ' is dense. Continuity of $\Psi(\theta, \phi; \xi)$ and compactness of the actual domain of interest, $[\theta_\ell, \theta_h] \times \digamma_\xi$, immediately imply that Ξ' is open, too. This establishes the claim.

Restrict the analysis to $\Xi' \subset \Xi$. For $\xi \in \Xi'$, consider a given pair (ϕ^*, ξ^*) and the (finite) collection of points $\theta_m \in G^{-1}(0)$. By labelling these values appropriately, we can write

$$\Theta^{0e} = \sum_{m=1}^{M} [\theta_m, \theta_{m+1}] + \sum_{m=M+1}^{M'} \theta_m \quad \text{and} \quad \mu\left(\Theta^{0e}\right) = \sum_{m=1}^{M} \mu\left([\theta_m, \theta_{m+1}]\right).$$

If $\frac{\partial G}{\partial \theta_m} \neq 0$, and $\frac{\partial G}{\partial \theta_{m+1}} \neq 0$, by the implicit function theorem (IFT from now on), (locally) there is a C^1 function $\theta_m(\phi; \xi)$ such that $G(\theta_m(\phi; \xi), \phi; \xi) = 0$, and $\mu([\theta_m(\phi; \xi), \theta_{m+1}(\phi; \xi)]) = |\theta_{m+1}(\phi; \xi) - \theta_m(\phi; \xi)|$ is clearly a continuous function, and there is nothing else to show.

If $\frac{\partial G}{\partial \theta_m} = 0$, by construction of Ξ', θ_m is either an inflexion point or a (locally unique) extremum of $G(\theta, \phi^*; \xi^*)$. Assume that it is an inflexion point and that $G(\theta, \phi^*; \xi^*)$ is increasing in θ at θ_m. Pick any sequence $\{(\phi^n; \xi^n)\}_{n=1}^{\infty}, (\phi^n; \xi^n) \to (\phi^*; \xi^*)$. First, we show that, for n large enough, there is a sequence $\{\theta^n\}_{n=n^*}^{\infty}$ such that $\theta^n \to \theta_m$ and $G(\theta^n, \phi^n; \xi^n) = 0$. Given that θ_m is an inflexion point, without loss of generality, assume that the function $G(\theta, \phi^*; \xi^*)$ is increasing in θ, at θ_m. Pick an open interval $\left(\underline{\theta}, \overline{\theta}\right)$, with $\theta_m \in \left(\underline{\theta}, \overline{\theta}\right)$, and sufficiently small, so that $G(\theta, \phi^*; \xi^*) < 0$ for each $\theta \in \left(\underline{\theta}, \theta_m\right)$, and $G(\theta, \phi^*; \xi^*) > 0$, for each $\theta \in \left(\theta_m, \overline{\theta}\right)$. By continuity of $G(\theta, \phi; \xi)$, for n large, $G(\overline{\theta}, \phi^n; \xi^n) > 0 > G(\underline{\theta}, \phi^n; \xi^n)$ and, therefore, by the intermediate value theorem, there is θ^n such that $G(\theta^n, \phi^n; \xi^n) = 0$. Clearly, it must be unique, because, for n sufficiently large, $G(\theta, \phi^n; \xi^n)$ is increasing in θ on the interval $\left(\underline{\theta}, \overline{\theta}\right)$. Hence, for some $\varepsilon > 0$, there is a function $\theta(\phi; \xi)$ such that $G(\theta(\phi; \xi), \phi; \xi) = 0$ for each $(\phi; \xi) \in V_\varepsilon(\phi^*; \xi^*)$. Continuity of $\theta(\phi; \xi)$ can be established taking a vanishing sequence of intervals $\left(\underline{\theta}, \overline{\theta}\right)$. It is also clear that, if $G(\theta_m, \phi^n; \xi^n) > G(\theta_m, \phi^*; \xi^*)$, $\theta(\phi^n; \xi^n) < \theta_m$, i.e., that $sign(\theta(\phi^n; \xi^n) - \theta_m) = -sign(G(\theta_m, \phi^n; \xi^n) - G(\theta_m, \phi^*; \xi^*))$.

Finally, consider the case of local extrema, which are (at most) a finite collection of isolated points. Pick a sufficiently small open ball $V_\varepsilon(\theta_m)$ (such that $\theta_m = V_\varepsilon(\theta_m) \cap G^{-1}(0)$) and any sequence $\{(\phi^n; \xi^n)\}_{n=1}^{\infty}, (\phi^n; \xi^n) \to (\phi^*; \xi^*)$. Assume, for instance, that θ_m is a local minimum of $G(\theta_m, \phi^*; \xi^*)$. Define as

$I(\phi^n;\xi^n) = conv\left(V_\varepsilon(\theta_m) \cap G^{-1}_{(\phi^n;\xi^n)}(0)\right)$, the interval obtained from the boundary points given by $V_\varepsilon(\theta_m) \cap G^{-1}_{(\phi^n;\xi^n)}(0)$. If $I(\phi^n;\xi^n) = \emptyset$, set $\mu(I(\phi^n;\xi^n)) = 0$. It is straightforward to check that, for n large, $V_\varepsilon(\theta_m) \cap G^{-1}_{(\phi^n;\xi^n)}(0)$ contains at most two points, and that $\mu(I(\phi^n;\xi^n)) \to 0$ as $(\phi^n;\xi^n) \to (\phi^*;\xi^*)$. Moreover, observe that $I(\phi^n;\xi^n) = \emptyset$, whenever $G(\theta_m,\phi^n;\xi^n) > G(\theta_m,\phi^*;\xi^*)$, while $I(\phi^n;\xi^n)$ is non trivial whenever $G(\theta_m,\phi^*;\xi^*) < G(\theta_m,\phi^n;\xi^n)$. Hence, $\mu(I(\phi^n;\xi^n)) > 0$ if and only if $sign(G(\theta_m,\phi^n;\xi^n) - G(\theta_m,\phi;\xi)) > 0$. Given that $\mu\left(I\left(\underset{-}{\theta},\bar{\theta}\right) \cap \Theta^{0e}(\phi^*;\xi^*)\right) = 0$, for $\left(\underset{-}{\theta},\bar{\theta}\right)$ small enough,

$$\begin{aligned}
& sign\left(\mu\left(\Theta^{0e}(\phi^n;\xi^n) \cap I\left(\underset{-}{\theta},\bar{\theta}\right)\right) - \mu\left(\Theta^{0e}(\phi^*;\xi^*) \cap I\left(\underset{-}{\theta},\bar{\theta}\right)\right)\right) \\
= \; & -sign(G(\theta_m,\phi^n;\xi^n) - G(\theta_m,\phi^*;\xi^*))
\end{aligned}$$

A similar argument can be used for local maxima. ■

Proof of Proposition 1. Consider the set $\Theta^{0e}(\phi;\xi)$.

If, at an interval $[\theta_m(\phi;\xi), \theta_{m+1}(\phi;\xi)] \subset \Theta^{0e}(\phi;\xi)$, $\frac{\partial G}{\partial \theta_m} \neq 0$ and $\frac{\partial G}{\partial \theta_{m+1}} \neq 0$, we can apply the IFT and $\frac{\partial \theta_m}{\partial \xi} = -\frac{\frac{\partial G}{\partial \xi}}{\frac{\partial G}{\partial \theta_m}}$. By definition (given that $\theta_m(\phi;\xi)$ is the lower bound of the interval), $\frac{\partial G}{\partial \theta_m} > 0$, while $\frac{\partial G}{\partial \theta_{m+1}} < 0$ at the upper bound. Hence, $sign\ \frac{\partial \mu([\theta_m(\phi;\xi),\theta_{m+1}(\phi;\xi)])}{\partial \xi} = sign \frac{\partial G}{\partial \xi}$. Evidently, the same argument holds for $\frac{\partial \mu([\theta_m(\phi;\xi),\theta_{m+1}(\phi;\xi)])}{\partial \phi^k}$. Taking into consideration the facts established in the proof of Lemma 1, essentially the same argument applies when some $\theta_m(\phi;\xi)$ is an inflexion point or an isolated local extremum: The sign of the change of the measure of the set $\Theta^{0e}(\phi;\xi)$ at θ_m is always equal to the sign of the relevant derivative $\frac{\partial G}{\partial \xi}$ (or $\frac{\partial G}{\partial \phi^k}$). Then, the Lemma immediately follows by direct computation of $\frac{\partial G}{\partial \xi}$ and $\frac{\partial G}{\partial \phi^k}$. ■

8.3. Appendix 3: Existence of SSE and of interior SSE

Here, and in the next Appendices, it is convenient to replace $\Phi(\phi,T;\xi)$ with

$$\begin{aligned}
\Phi'(\phi,T;\xi) &\equiv \begin{pmatrix} \frac{r+\beta\pi^e(\phi^e)}{1-\beta}\Phi^e(\phi,T;\xi) \\ \frac{r+\beta\pi^{ne}(\phi^{ne})}{1-\beta}\Phi^{ne}(\phi,T;\xi) \end{pmatrix} \equiv \begin{pmatrix} F^e(\phi,T;\xi) - A^e(\phi^e;\xi) \\ F^{ne}(\phi,T;\xi) - A^{ne}(\phi^{ne};\xi) \end{pmatrix} \\
&\equiv \begin{pmatrix} \frac{\int_{L^{e\alpha}(\phi)}(f^e(\vartheta)-b^e(\vartheta))d\vartheta}{\mu(L^{e\alpha}(\phi))} - v^e\frac{r+\beta\pi^e(\phi^e)}{(1-\beta)q^e(\phi^e)} \\ \frac{\int_{L^{ne\alpha}(\phi)}(f^{ne}(\vartheta)-b^{ne}(\vartheta))d\vartheta}{\mu(L^{ne\alpha}(\phi))} - v^{ne}\frac{r+\beta\pi^{ne}(\phi^{ne})}{(1-\beta)q^{ne}(\phi^{ne})} \end{pmatrix}.
\end{aligned}$$

Clearly, $\Phi(\phi,T;\xi) = 0$ if and only if $\Phi'(\phi,T;\xi) = 0$. This transformation is convenient because, for each k, $F^k(\phi,T;\xi)$ depends on ϕ only trough its effect on $\Theta^e(\phi)$, while $A^k(\phi^k;\xi)$ only depends on ϕ^k. Bear in mind that Θ^k are at their steady state values.

8.3.1. *Existence of SSE*

Proof of Theorem 1. In the proof, $(T;\xi)$ is fixed. Hence, omitted from the notation. Let

$$\begin{aligned} \underline{\phi}^k &= \frac{1}{2}\left\{\phi^k | A^k(\phi^k) = \min_\theta \left(f^k(\theta) - b^k(\theta)\right)\right\}, \\ \bar{\phi}^k &= 2\left\{\phi^k | A^k(\phi^k) = \max_\theta \left(f^k(\theta) - b^k(\theta)\right)\right\}. \end{aligned}$$

By Ass. 1, $\min_\theta \left(f^k(\theta) - b^k(\theta)\right) > 0$, for each k. By compactness of $[\theta_\ell, \theta_h]$ and continuity of (f, b), there is $\theta \in \max_\theta \left(f^k(\theta) - b^k(\theta)\right)$. For each k, the function $A^k(\phi^k)$ is easily seen to be continuous and strictly increasing. Moreover, given that $A^k(\phi^k) = \frac{v^k}{1-\beta}\left(\frac{r}{q^e(\phi^e)} + \beta\phi^e\right)$, and Ass. 2, $A^k(\mathbb{R}_{++}) = \mathbb{R}_{++}$. Hence, $(\underline{\phi}^k, \bar{\phi}^k)$ exists and is unique, for each k. Evidently, if a SSE ϕ^* exists, it must be $\phi^* \in int \prod_k [\underline{\phi}^k, \bar{\phi}^k]$, a compact, convex, non-empty subset of $\mathbb{R}^2_{++}$.

Consider a modified economy where, for each θ, there is an interval $(1+\delta)$ of agents of ability θ. Assume that, for each θ, a measure δ of individuals always invest in education, while the remaining individuals choose to invest or not as in the actual economy. Let Ω^k be the counterpart of the set L^k in this modified economy. Restrict the analysis to the set of economies Ξ' such that $\mu(L^e(\phi))$ is a continuous function (see Lemma 1). Given $\xi \in \Xi'$, at each ϕ, and taking into account (2) above, at a SSE, $L^e(\phi)$ is either empty or

$$L^e(\phi) = \left\{\theta \in \Theta | \left[\bigcup_{m=1}^{M'} [\theta_{im}, \theta_{im+1}] \bigcup_{m=M'+1}^{M} \theta_{im}\right], \text{ each } i \le e^{-\gamma T}\alpha(\theta).\right\}$$

We have already shown (see the proof of Lemma 1) that each non trivial interval is locally a continuous correspondence of ϕ. Let $\mu(\emptyset) = 0$. Evidently,

$$\begin{aligned} F^e_\delta(\phi) \equiv & \frac{\int_{\Omega^e(\phi)} \left(f^e(\vartheta) - b^e(\theta)\right) d\vartheta}{\mu(\Omega^e(\phi))} = \frac{e^{-\gamma T}\sum_{m=1}^{M'} \int_{\theta_m(\phi)}^{\theta_{m+1}(\phi)} \alpha(\vartheta)\left(f^e(\vartheta) - b^e(\theta)\right) d\vartheta}{\mu(\Omega^e(\phi))} \\ & + \frac{e^{-\gamma T}\delta \int_{\theta_\ell}^{\theta_h} \alpha(\vartheta)\left(f^e(\vartheta) - b^e(\theta)\right) d\vartheta}{\mu(\Omega^e(\phi))} \\ & + \frac{e^{-\gamma T}\sum_{m=M'+1}^{M} \int_{\theta_m(\phi)}^{\theta_m(\phi)} \alpha(\vartheta)\left(f^e(\vartheta) - b^e(\theta)\right) d\vartheta}{\mu(\Omega^e(\phi))}. \end{aligned}$$

For each $\delta > 0$, $\mu(\Omega^e(\phi)) \ge e^{-\gamma T}\frac{\delta \int_{\theta_\ell}^{\theta_h} \alpha(\vartheta) d\vartheta}{1+\delta} > 0$. Hence, the first two terms of $F^e_\delta(\phi)$ are continuous functions of ϕ. The second term (by definition and the assumption $(f^e(\vartheta) - b^e(\theta)) > 0$, for each θ) is bounded away from zero.

Consider the last term. Pick any ϕ^*. For each isolated point θ_m, $m > M'$, let $V(\theta_m)$ be any open ball such that $V(\theta_m) \cap G^{-1}_{\phi^*}(0) = \theta_m$. Pick any $\varepsilon > 0$. For each sequence $\{\phi^n\}_{n=1}^\infty$, $\phi^n \to \phi^*$, for n large enough, $\mu(V(\theta_m) \cap G^{-1}_{\phi^n}(0)) \le \varepsilon$, i.e., along any such a sequence, $\mu(V(\theta_m) \cap G_{\phi^n}^{-1}(0))$ is either 0, or vanishing. This immediately implies that $F_\delta^e(\phi)$ is a continuous function of ϕ.

Given that $\mu(\Omega^{ne}(\phi)) \geq \frac{e^{-\gamma T}}{1+\delta} \int_{\theta_\ell}^{\theta_h} (1-\alpha(\vartheta))\, d\vartheta > 0$, a similar argument shows that $F_\delta^{ne}(\phi)$ is also a continuous function.

Pick $\delta > 0$. Let $\phi(F_\delta)$ be the solution to $A^k(\phi^k) = F_\delta^e$. Given $(T;\xi)$, strict monotonicity of $A^k(\phi^k)$, for each k, implies that a solution exists, is unique and is described by a continuous function, call it $\phi(F_\delta)$. Given that $F_\delta^k(\phi)$ is continuous, each k, $\phi(F_\delta(\phi))$ maps $\prod_k [\underline{\phi}^k, \bar{\phi}^k]$ continuously into itself. Hence, by Brower's fixed point theorem, there is ϕ^δ such that $\phi^\delta = \phi(F_\delta(\phi^\delta))$.

Consider a sequence $\{\delta^n\}_{n=1}^\infty$, with $\delta^n > 0$, each n and $\delta^n \to 0$ and the associated sequence of fixed points $\left\{\phi^{\delta^n}\right\}_{n=1}^\infty \subset \prod_k [\underline{\phi}^k, \bar{\phi}^k]$. Compactness of this set implies that $\left\{\phi^{\delta n}\right\}_{n=1}^\infty$ contains a convergent subsequence with limit $\phi^* \in \prod_k [\underline{\phi}^k, \bar{\phi}^k]$. If $L^e(\phi^*) \neq \emptyset$, it is easy to check that ϕ^* is a SSE of the actual economy. Otherwise, ϕ^* is still a SSE, because (up to a renormalization) $\{F_\delta(\phi^{\delta n})\}_{n=1}^\infty$ provides us with the sequence required by the definition of equilibrium, when we cannot apply Bayes' rule. ■

8.3.2. *Existence of interior SSE*

Proof of Lemma 2. Consider the artificial economy with fixed set $\Theta^{0e} = \Theta^0$. The two labor markets are then independent and, evidently, in each one there is a unique SSE ϕ^*. We just need to show that, for T small enough, this is a SSE of the actual economy, i.e., that $G(\theta, \phi^*, T) \geq 0$, for each θ. By assumption, $F^e(\phi^{e*}, T) > F^{ne}(\phi^{ne*}, T)$, while $v^e = v^{ne}$ and $q^e(\phi) = q^{ne}(\phi)$. Therefore, $\phi^{e*} > \phi^{ne*}$, because $\frac{\partial A^k}{\partial \phi^k} > 0$, for each k.

Consider eq. (8). Under Assumption 1, if $\phi^e = \phi^{ne}$, the term in square brackets is positive. By direct computation, for each θ, $\frac{\partial G}{\partial \phi^e} = \frac{\beta r (f^e(\theta) - b^e(\theta))}{(r + \beta\pi(\phi^e))^2} \frac{\partial \pi(\phi^e)}{\partial \phi^e} > 0$, because $(f^e(\theta) - b^e(\theta)) > 0$, each θ, and $\frac{\partial \pi(\phi^e)}{\partial \phi^e} > 0$. By assumption, $\alpha(\theta) > 0$, for each θ. Hence, the first term of $G(\theta, \phi^*, T)$ is strictly positive for each θ. For $T = 0$, the second term is nil. Therefore, by continuity, for T small enough, $G(\theta, \phi^*; T) \geq 0$ for each θ, and ϕ^* is a SSE. Evidently, for T large enough, $G(\theta, \phi^*; T) < 0$ for each θ. Hence, the set of values of $T \in \mathbb{R}_{++}$ such that the given pair ϕ^* is a SSE (and $\Theta^e(\phi^*, T) = \Theta$) is bounded above. Given that $G(\theta, \phi^*; T)$ is continuous in T, there is

$$T^* = \max\left\{T \in (0, \infty) \,|\phi^* \text{ is a SSE at } T \text{ and } \Theta^{0e}(\phi^*) = \Theta\right\},$$

and ϕ^* is a SSE if and only if $T \leq T^*$. ■

Given an economy $\xi \in \Xi$, let ϕ_ξ be the SSE for $\Theta^e = \Theta$, and T_ξ be the maximum value of T such that ϕ_ξ is a SSE of the economy. By Assumption 3, T_ξ exists.

We split the proof of the Theorem into several steps. First, we show that, for a generic set of economies, at the associated T^* (as defined in Lemma 2) $G^{-1}(0)$ is a discrete set. The proof is slightly different from the one of Lemma 1, also because T is an additional variable, now. In the second step, we fix the pair (ϕ_ξ, T_ξ) of a given ξ. We show that there is an economy ξ' (arbitrarily close to ξ) and some $T_{\xi'} > T_\xi$ such that ϕ_ξ is an interior SSE of the new economy. The third step is to show that $\text{rank} D_\phi \Phi(\phi_\xi, T_{\xi'}; \xi') = 2$. The Theorem then follows by the IFT.

Fact 1. For an open, dense subset $\Xi' \subset \Xi$, $G^{-1}(0)$ is a discrete set at (ϕ_ξ, T_ξ), for each $\xi \in \Xi'$.

Proof of Fact 1.

Fix $\Theta^e = \Theta$. Consider the map $\Psi : [\theta_\ell, \theta_h] \times \mathbb{R}^3_{++} \times \Xi \to \mathbb{R}^5$, defined by

$$\Psi(\theta, \phi; T, \xi) = \begin{pmatrix} \Phi'(\theta, \phi; T, \xi) \\ G(\theta, \phi; T, \xi) \\ \frac{\partial G(\theta,\phi;T,\xi)}{\partial \theta} \\ \frac{\partial^2 G(\theta,\phi;T,\xi)}{\partial \theta^2} \end{pmatrix},$$

and replace the function $c(\theta)$ with

$$c(\theta; d) = c(\theta) + d_0^c + d_1^c \theta + d_2^c \theta^2,$$

where $d \in D$, a sufficiently small, open subset of $\mathbb{R}^3$, with $0 \in D$. Assume that $\Psi \pitchfork 0$. Then, by TT, except for a null subset of Ξ, $\Psi_\xi \pitchfork 0$. Given that $\Psi_\xi : [\theta_\ell, \theta_h] \times \mathbb{R}^3_{++} \to \mathbb{R}^5$, this implies that $\Psi_\xi^{-1}(0) = \emptyset$, i.e., that, whenever $\Phi'_\xi(\theta, \phi; T) = 0$, at each $\theta \in G_\xi^{-1}(0)$, $\frac{\partial^n G_\xi(\theta,\phi;T)}{\partial \theta^n} \neq 0$, for at least one $n \in \{1, 2\}$. This means that each $\theta \in G_\xi^{-1}(0)$ is neither a degenerate local extremum, nor an inflexion point. To show that $\Psi \pitchfork 0$, consider $D_{(v,d)} \Psi(\theta, \phi; T, \xi) =$

$$\begin{array}{ccccc} v^e & v^{ne} & d_0^c & d_1^c & d_2^c \end{array}$$
$$\begin{bmatrix} -\frac{r+\beta\pi^e(\phi^e)}{(1-\beta)q^e(\phi^e)} & 0 & 0 & 0 & 0 \\ 0 & -\frac{r+\beta\pi^e(\phi^e)}{(1-\beta)q^e(\phi^e)} & 0 & 0 & 0 \\ 0 & 0 & \left(1-e^{rT}\right) & \left(1-e^{rT}\right)\theta & \left(1-e^{rT}\right)\theta^2 \\ 0 & 0 & 0 & \left(1-e^{rT}\right) & 2\left(1-e^{rT}\right)\theta \\ 0 & 0 & 0 & 0 & 2\left(1-e^{rT}\right) \end{bmatrix},$$

which is obviously of full rank 5. Hence, by TT, except for a null subset of D, D^1, $\Psi_\xi \pitchfork 0$. Restricting the analysis to $\phi \in \digamma_\xi$, and T lying in any compact subset of $\mathbb{R}_+$, we obtain that $D \backslash D^1$ is open and dense. Going from polynomial perturbations to the set Ξ, we have established that, for an open, dense subset of $\Xi, G^{-1}(0)$ is a discrete set at each (ϕ_ξ, T_ξ).

Pick an economy $\xi \in \Xi'$. Let ϕ_ξ the associate SSE at T_ξ. Evidently, for $\varepsilon > 0$ sufficiently small,

a) if $\theta_\ell \in G^{-1}(0)$, $\frac{\partial G\left(\theta,\phi_\xi,T_\xi;\xi\right)}{\partial\theta}|_{\theta\in(\theta_\ell,\theta_\ell+\varepsilon)} > 0$;

b) if $\theta_h \in G^{-1}(0)$, $\frac{\partial G\left(\theta,\phi_\xi,T_\xi;\xi\right)}{\partial\theta}|_{\theta\in(\theta_h-\varepsilon,\theta_h)} < 0$;

c) if $\theta' \in G^{-1}(0) \cap (\theta_\ell, \theta_h)$, $\frac{\partial G\left(\theta,\phi_\xi,T_\xi;\xi\right)}{\partial\theta}|_{\theta\in(\theta'-\varepsilon,\theta')} < 0$ and $\frac{\partial G\left(\theta,\phi_\xi,T_\xi;\xi\right)}{\partial\theta}|_{\theta\in(\theta',\theta'+\varepsilon)} > 0$, i.e., θ' is a local minimum.

Fact 2. Under assumptions 1-3, there is a dense subset $\Xi'' \subset \Xi'$ such that, for each $\xi \in \Xi''$, there is an interior SSE $\phi(T)$, for some T. Moreover, at such a SSE, $\frac{\partial G(\theta,\phi(T),T;\xi))}{\partial\theta}|_{\theta_m} \neq 0$, at each $\theta_m \in G^{-1}(0)$.

Proof of Fact 2. Take $\xi \in \Xi'$. Fix ϕ_ξ and pick any $T > T_\xi$. Remember that the function $G\left(\theta, \phi_\xi, T; \xi\right)$ is strictly decreasing in T. Therefore,

$$\Theta^{0e}(\phi_\xi, T; \xi) \equiv \left\{\theta \in \Theta^0 | G\left(\theta, \phi_\xi, T; \xi\right) \geq 0\right\}$$

is a proper subset of Θ^0, and, by continuity, nonempty, for each T sufficiently close to T_ξ, $T > T_\xi$. Moreover, for $T > T_\xi$ and sufficiently close to T_ξ, $\# G^{-1}_{(\phi_\xi, T; \xi)}(0) \leq 2\# G^{-1}_{(\phi_\xi, T^*_\xi; \xi)}(0)$ (so that $G^{-1}_{(\phi_\xi, T; \xi)}(0)$ is a discrete set), and $\frac{\partial G\left(\theta, \phi_\xi, T; \xi\right)}{\partial \theta} \neq 0$, for each $\theta \in G^{-1}(0)$. Also, we can pick T such that $G\left(\theta_\ell, \phi_\xi, T; \xi\right) \neq 0$ and $G\left(\theta_h, \phi_\xi, T; \xi\right) \neq 0$. Finally, observe that $(a - c)$ above imply that the correspondence $\Theta^e(\phi_\xi, T; \xi)$ is continuous in T along sequences $\{T^n\}$, $T^n \geq T_\xi$ each n (notice that, when at $\left(\phi_\xi, T_\xi\right)$, $G^{-1}(0)$ contains an interval, we lose this property). Evidently the properties just established hold for each ξ in some open, dense set $\Xi'' \subset \Xi'$: at each $T > T_\xi$, T close enough to T_ξ, $\Theta^{0e}(\phi_\xi, T; \xi) \neq \Theta^0$ and $\Theta^{0e}(\phi_\xi, T; \xi) \neq \emptyset$. Evidently, the given ϕ_ξ is a SSE at T_ξ, and it is not necessarily a SSE at $T > T_\xi$.

We now perturb the parameter v so that, in the new economy ξ', ϕ_ξ is a SSE at some $T > T_\xi$. Given that $G(\theta, \phi, T; \xi)$ does not depend upon v, changes in this parameter have no effect on the set $\Theta^{0e}(\phi_\xi, T; \xi)$. It is easy to check that $F^k\left(\phi_\xi, T; \xi\right)$ is a continuous function of $(T; \xi)$ (given that $\mu(L^k(\phi_\xi, T; \xi))$ is locally bounded away from zero). Hence, for $T \to T_\xi$, $\Phi'\left(\phi_\xi, T; \xi\right) \to 0$. Given that $A^k\left(\phi_\xi, T; \xi\right) \neq 0$ (and is $T-$invariant), given any $\varepsilon > 0$, for T sufficiently close to T_ξ there is $v' \in B_\varepsilon(v)$ such that $\Phi'(\phi_\xi, T; \xi^\setminus, v') = 0$. ■

Proof of Theorem 2. Pick $\xi \in \Xi$", and T_ξ such that, at the associated interior SSE, $\frac{\partial G(\theta, \phi(T), T; \xi))}{\partial \theta}|_{\theta_m} \neq 0$, at each $\theta_m \in G^{-1}(0) \cap (\theta_\ell, \theta_h)$, $G\left(\theta_\ell, \phi\left(T\right), T; \xi\right)) \neq 0$ and $G\left(\theta_h, \phi\left(T\right), T; \xi\right)) \neq 0$. Then, $D_\phi \Phi(.)$ exists and is continuous. Hence, by the IFT, there is a collection of C^1 functions, $\theta_m\left(\phi, T; \xi\right)$ locally describing the set $G^{-1}(0)$, i.e., the boundary points of the set $L^e\left(\phi, T; \xi\right)$. Therefore, $\Phi(\phi, T; \xi)$ is C^1.

All we need to conclude is to show that these properties imply that there is also a dense subset $\Xi^\circ \subset \Xi$ such $D_\phi \Phi(\phi, T; \xi)$ has full rank at an interior SSE.

Pick any economy ξ'' in the dense set Ξ'' constructed in Fact 2. Let ϕ'' be the associated interior SSE at T''.

The strategy of the proof is the following: First, we show that (locally) arbitrary changes in the pair $\left(v^e, v^{ne}\right)$, call them $d_v = \left(d^e_v, d^{ne}_v\right)$, can be compensated by appropriate changes in the production functions, call them $d_f = \left(d^e_f\left(d_v\right), d^{ne}_f\left(d_v\right)\right)$, so that ϕ'' is still a SSE in the new economy at T''. Next, we show that we can pick a d_v arbitrarily small and such that, in the new economy, ξ°, arbitrarily close to ξ'', $\det D_\phi \Phi_{d_v}(\phi, T''; \xi^\circ) \neq 0$ at ϕ''. By definition of density, this implies that there is a dense subset $\Xi^\circ \subset \Xi''$ such that, for each $\xi^\circ \in \Xi^\circ$, there is an interior SSE at some T_{ξ° with $\det D_\phi \Phi(\phi, T_{\xi^\circ}; \xi^\circ) \neq 0$. By iterated application of IFT, there is an open ball $V(\xi^\circ)$ such that, for each $\xi' \in V(\xi^\circ)$, there is an open ball $V(T_{\xi'})$ such that, at each $T \in V(T_{\xi'})$, the economy ξ' has an interior SSE with non-zero determinant. This proves the theorem.

To conclude the proof, we need to construct an economy ξ° with the stated properties, for each $\xi \in \Xi''$. We start defining the (finite dimensional) parameterization of the production functions.

Pick θ^* such that $G(\theta^*, \phi_\xi, T_\xi; \xi) > 0$, and, as above, pick some open ball $V_\varepsilon(\theta^*)$ such that, for each $\theta \in clV_{2\varepsilon}(\theta^*), G(\theta, \phi_\xi, T_\xi; \xi) > 0$, and a smooth bump function $\psi(\theta)$. For each k, define $f^k(\theta; d) = f(\theta) + \psi(\theta) d_f^k$. By continuity, for ε sufficiently small, this perturbation has no effect on the sets $L^e(\phi_\xi, T_\xi; \xi)$ and $L^{ne}(\phi_\xi, T_\xi; \xi)$. On the other hand, its effect on the value of $\Phi(\phi_\xi, T_\xi; \xi)$ is

$$\begin{bmatrix} \Delta_f \Phi'^e \\ \Delta_f \Phi'^{ne} \end{bmatrix} = \begin{bmatrix} \frac{\int_{V_{2\varepsilon}(\theta^*)} \alpha(\theta)\psi(\vartheta) d\vartheta}{\mu(L^e)} d_f^e \\ \frac{\int_{V_{2\varepsilon}(\theta^*)} (1-\alpha(\theta))\psi(\vartheta) d\vartheta}{\mu(L^{ne})} d_f^{ne} \end{bmatrix}.$$

Evidently,

$$\begin{bmatrix} \Delta_v \Phi'^e \\ \Delta_v \Phi'^{ne} \end{bmatrix} = \begin{bmatrix} -\frac{A^e(\phi^e)}{v^e} d_v^e \\ -\frac{A^{ne}(\phi^{ne})}{v^{ne}} d_v^{ne} \end{bmatrix}.$$

Hence, to preserve the equilibrium, it must be

$$\begin{bmatrix} d_f^e(d_v^e) \\ d_f^{ne}(d_v^{ne}) \end{bmatrix} = \begin{bmatrix} \frac{A^e(\phi^e)\mu(L^e)}{v^e \int_{V_{2\varepsilon}\left(\theta^\circ\right)} \alpha(\vartheta)\psi(\vartheta) d\vartheta} d_v^e \\ \frac{A^{ne}(\phi^{ne})\mu(L^{ne})}{v^{ne} \int_{V_{2\varepsilon}\left(\theta^\circ\right)} (1-\alpha(\vartheta))\psi(\vartheta) d\vartheta} d_v^{ne} \end{bmatrix}.$$

Let's define

$$\begin{aligned} \Phi'((\phi;\xi), d_f, d_v) &\equiv \begin{pmatrix} F^e(.) - A^e(.) + \Delta_f \Phi^{e\prime} d_f^e - \Delta_v \Phi'^e d_v^e \\ F^{ne}(.) - A^{ne}(.) + \Delta_f \Phi^{ne\prime} d_f^{ne} - \Delta_v \Phi'^{ne} d_v^{ne} \end{pmatrix} \\ &\equiv \Phi'(\phi;\xi) + \Delta(d_f, d_v). \end{aligned}$$

By direct computation,

$$D_\phi \Delta(d_f, d_v) \equiv - \begin{bmatrix} \Delta_f \Phi^{e\prime} \frac{\partial \mu(L^e)}{\partial \phi^e} + \frac{1}{v^e} \frac{\partial A^e}{\partial \phi^e} d_v^e & \Delta_f \Phi^{e\prime} \frac{\partial \mu(L^e)}{\partial \phi^{ne}} \\ \Delta_f \Phi^{ne\prime} \frac{\partial \mu(L^{ne})}{\partial \phi^e} & \Delta_f \Phi^{ne\prime} + \frac{1}{v^{ne}} \frac{\partial A^{ne}}{\partial \phi^{ne}} d_v^{ne} \end{bmatrix}$$

and, substituting into it the vector $d_f(d_v)$,

$$D_\phi \Delta(d_f(d_v), d_v) \equiv - \begin{bmatrix} \left(\frac{A^e(\phi^e)}{\mu(L^e) v^e} \frac{\partial \mu(L^e)}{\partial \phi^e} + \frac{1}{v^e} \frac{\partial A^e}{\partial \phi^e} \right) d_v^e & \frac{A^e(\phi^e)}{v^e \mu(L^e)} \frac{\partial \mu(L^e)}{\partial \phi^{ne}} d_v^e \\ \frac{A^{ne}(\phi^{ne})}{\mu(L^{ne}) v^{ne}} \frac{\partial \mu(L^{ne})}{\partial \phi^e} d_v^{ne} & \left(\frac{A^{ne}(\phi^{ne})}{\mu(L^{ne}) v^{ne}} \frac{\partial \mu(L^{ne})}{\partial \phi^{ne}} + \frac{1}{v^{ne}} \frac{\partial A^{ne}}{\partial \phi^{ne}} \right) d_v^{ne} \end{bmatrix},$$

where $\frac{\partial A^k}{\partial \phi^k} = \frac{1}{1-\beta} \frac{q^k(\phi^k) \frac{\partial \pi^k}{\partial \phi^k} - (r + \beta \pi^e(\phi^e)) \frac{\partial q^k}{\partial \phi^k}}{q^k(\phi^k)^2} > 0$, for each k. Given the results above, without any loss of generality, $\frac{\partial G}{\partial \theta_m} \neq 0$, at each $\theta_m \in G^{-1}(0)$. Then, by direct computation,

$$\mu(L^e(\phi)) = e^{-\gamma T} \sum_{m=1}^{M} \int_{\theta_m}^{\theta_{m+1}} \alpha(\vartheta) d\vartheta,$$

while

$$\mu(L^{ne}(\phi)) = \left(1 - \sum_{m=1}^{M} \int_{\theta_m}^{\theta_{m+1}} d\vartheta \right) + e^{-\gamma T} \sum_{m=1}^{M} \int_{\theta_m}^{\theta_{m+1}} (1 - \alpha(\vartheta)) d\vartheta.$$

Hence, using the function $\chi(\theta_m)$, such that $\chi(\theta_m) = 1$ if θ_m is the lower bound of an interval $[\theta_m, \theta_{m+1}] \subset \Theta^{0e}$, $\chi(\theta_m) = 2$ otherwise,

$$\frac{\partial \mu(L^e(\phi))}{\partial \phi^k} = e^{-\gamma T} \sum_{m=1}^{M} \alpha(\theta_m) (-1)^{\chi(\theta_m)} \frac{\partial \theta_m}{\partial \phi^k},$$

and

$$\frac{\partial \mu\left(L^{ne}(\phi)\right)}{\partial \phi^k} = e^{-\gamma T} \sum_{m=1}^{M} \left(1 - e^{\gamma T} - \alpha\left(\theta_m\right)\right)(-1)^{\chi(\theta_m)} \frac{\partial \theta_m}{\partial \phi^k}.$$

Then, it is easy to check that

$$\frac{\partial \mu\left(L^{e}(\phi)\right)}{\partial \phi^e} \frac{\partial \mu\left(L^{ne}(\phi)\right)}{\partial \phi^{ne}} = \frac{\partial \mu\left(L^{e}(\phi)\right)}{\partial \phi^{ne}} \frac{\partial \mu\left(L^{ne}(\phi)\right)}{\partial \phi^e}.$$

Moreover, $\frac{\partial \mu\left(L^{k}(\phi)\right)}{\partial \phi^k} > 0$, $\frac{\partial \mu\left(L^{k}(\phi)\right)}{\partial \phi^{k'}} < 0$. Hence, taking into account the results above,

$$\frac{v^e v^{ne}}{d_v^e d_v^{ne}} \det D_\phi \Delta\left(.\right) = \frac{\partial A^e}{\partial \phi^e} \frac{\partial A^{ne}}{\partial \phi^{ne}} + \frac{\partial A^e}{\partial \phi^e} \frac{A^{ne}\left(\phi^{ne}\right)}{\mu\left(L^{ne}\right)} \frac{\partial \mu\left(L^{ne}\right)}{\partial \phi^e} + \frac{\partial A^{ne}}{\partial \phi^{ne}} \frac{\partial \mu\left(L^{e}\right)}{\partial \phi^e} \frac{A^{e}\left(\phi^{e}\right)}{\mu\left(L^{e\alpha}\right)} > 0.$$

We conclude showing that, for an appropriate choice of d_v, this implies that

$$\det D_\phi \Phi'\left((\phi;\xi), d_f, d_v\right) \neq 0.$$

Indeed, consider any matrix

$$B = \begin{bmatrix} a - a_1 d_v^e & b - b_1 d_v^e \\ c - c_1 d_v^{ne} & e - e_1 d_v^{ne} \end{bmatrix},$$

with $\det B = (ae - bc) + (bc_{\emptyset 1} - ae_1)\, d_v^{ne} + (cb_1 - ea_1)\, d_v^e + (a_1 e_1 - b_1 c_1)\, d_v^e d_v^{ne}$. Assume that $(ae - bc) = 0$ (otherwise there is nothing to show) and that $(a_1 e_1 - b_1 c_1) \neq 0$. If $(bc_1 - ae_1) = (cb_1 - ea_1) = 0$, there is nothing else to show. Otherwise, pick any d_v^e such that $\left[(bc_1 - ae_1) + (a_1 e_1 - b_1 c_1)\, d_v^e\right] \neq 0$ (this can be done because $(a_1 e_1 - b_1 c_1) \neq 0$). Then, $\det B \neq 0$ whenever

$$d_v^{ne} \neq \frac{-\left(cb_1 - ea_1\right) d_v^e}{\left(bc_1 - ae_1\right) + \left(a_1 e_1 - b_1 c_1\right) d_v^e}.$$

Using the notation introduced above, $\det D_\phi \Delta\left(d_f(d_v), d_v\right) \neq 0$ means $(a_1 e_1 - b_1 c_1) \neq 0$. Then, we just pick a pair (d_v^e, d_v^{ne}) satisfying the last two inequalities (so that $\det D_\phi \Phi'\left((\phi;\xi), d_f, d_v\right) \neq 0$) and sufficiently small, so that the economy $\overset{\circ}{\xi}$ so obtained is sufficiently close to the original economy ξ. ■

8.3.3. Generic regularity of SSE

Proof of Theorem 3. Evidently, $\Xi_T \neq \emptyset$.

Assume that Ξ_T^{reg} is not dense in Ξ_T, then we can find an open (relative to Ξ_T) set $V\left(\xi^\circ\right) \subset \Xi_T \backslash \Xi_T^{reg}$. We start showing that there is a residual (hence, dense) subset of $V\left(\xi^\circ\right) \subset \Xi_T^{reg}$.

Fix $N \in \mathbb{N}$. Pick any collection of N distinct elements of (θ_ℓ, θ_h) with rational coordinates, $\theta_N = \{\theta_1, ...\theta_n, ..., \theta_N\}$. Define

$$\varepsilon = \min\left\{\min_{n,n'} dist\left(\theta_n, \theta_{n'}\right), \min_n dist\left(\theta_n, \theta_h\right), \min_n \left(\theta_n, \theta_\ell\right)\right\}.$$

Evidently, $\varepsilon > 0$, and $clV_{\frac{2\varepsilon}{3}}\left(\theta_n\right) \cap clV_{\frac{2\varepsilon}{3}}\left(\theta_{n'}\right) = \emptyset$, for each pair θ_n, $\theta_{n'}$, and θ_ℓ, $\theta_h \notin clV_{2\frac{\varepsilon}{3}}\left(\theta_n\right)$, for each n. Consider all the possible partitions of the collection θ_N into two (possibly empty) sets, call them $P_s \in$ Þ. Evidently, the cardinality of Þ is finite for each N. Pick a partition $P_s \equiv \{P_s^r, P_s^0\} \in$Þ$_s$, $P_s \in$Þ. Without any loss of generality assume that (modulo a relabelling) $P_s^r = \{\theta_1, ..., \theta_{NR}\}$ and $\theta_1 < ... < \theta_{NR}$ If $\# P_s^r$ is even, define:

a. $\Theta^{0e1}(P_s^r) = [\theta_\ell, \theta_1] \cup [\theta_2, \theta_3] \cup .... \cup [\theta_{NR}, \theta_h]$,

b. $\Theta^{0e2}(P_s^r) = [\theta_1, \theta_2] \cup [\theta_3, \theta_4] \cup .... \cup [\theta_{NR-1}, \theta_{NR}]$.

If it is odd, define:

c. $\Theta^{0e3}(P_s^r) = [\theta_\ell, \theta_1] \cup [\theta_2, \theta_3] \cup .... \cup [\theta_{NR-1}, \theta_{NR}]$,

d. $\Theta^{0e4}(P_s^r) = [\theta_1, \theta_2] \cup [\theta_3, \theta_4] \cup .... \cup [\theta_{NR}, \theta_h]$.

Use ζ, $\zeta = 1, ..., 4$, to refer to the indexes above. Redefine the map $\Phi^{e'}(\theta, \phi; \xi)$ as $\Phi^{e'}(\theta, \phi; P_s^r, \zeta, \xi) = \left(\frac{\int_{\Theta^{0e\zeta}(P_s^r)} \alpha(\theta)(f^e(\vartheta) - b^e(\vartheta)) d\vartheta}{\mu(\Theta^{0e\zeta}(P_s^r))} - A^e(\phi^e; \xi) \right)$, for each ζ. Redefine $\Phi^{ne'}(\theta, \phi; P_s^r, \zeta, \xi)$ in a similar way. Set $\theta^a = [\theta_1, ..., \theta_{NR}]$ and $\theta^b = [\theta_{NR+1}, ..., \theta_N]$. Define the maps,

$$\Phi^E(\theta^a, \phi; P_s^r, \zeta, \xi) = \begin{bmatrix} \Phi'(\theta^a, \phi; P_s^r, \zeta, \xi) \\ G(\theta_n, \phi; P_s^r, \zeta, \xi) \quad \text{for } n \in P_s^r \end{bmatrix},$$

$$\Psi(\theta, \phi; P_s^r, \zeta, \xi) = \begin{bmatrix} \Phi^E(\theta^a, \phi; P_s^r, \zeta, \xi) \\ G(\theta_n, \phi; P_s^r, \zeta, \xi) \quad \text{for } n \in P_s^0 \\ \frac{\partial G(\theta, \phi; P_s^r, \zeta, \xi)}{\partial \theta}|_{\theta = \theta_n} \quad \text{for } n \in P_s^0 \end{bmatrix},$$

and

$$\Phi^{Ey}(\theta, \phi; P_s^r, \zeta, \xi) = \begin{bmatrix} \Phi^E(\theta^a, \phi; P_s^r, \zeta, \xi) \\ G(\theta_y, \phi; P_s^r, \zeta, \xi) \end{bmatrix}, \quad \text{for} \quad y = \ell, h.$$

Notice that (θ', ϕ') is an interior SSE if and only if $\Phi^E(\theta^{a'}, \phi'; P_s^r, \zeta, \xi) = 0$ and the associated set $\Theta^{0e\zeta}(P_s^r)$ coincides a.e. with the actual set $\Theta^{0e}(\phi')$. An important difference with respect to the map $\Phi'(\theta, \phi; \xi)$ considered above is that $\Phi'(\theta, \phi; \xi)$ may fail to be C^1 (because $\frac{\partial \theta}{\partial \phi}$ may not exist at some $\theta \in G^{-1}(0)$), while $\Psi^E(\theta', \phi'; P_s^r, \zeta, \xi)$ is always C^1. Also notice that

$$\Phi^E : clV_{\frac{\varepsilon}{3}}(\theta_1) \times ... \times clV_{\frac{\varepsilon}{3}}(\theta_{NR}) \times F_{\xi^\circ} \times V(\xi^\circ) \to \mathbb{R}^{2+NR},$$

$$\Psi : clV_{\frac{\varepsilon}{3}}(\theta_1) \times ... \times clV_{\frac{\varepsilon}{3}}(\theta_N) \times F_{\xi^\circ} \times V(\xi^\circ) \to \mathbb{R}^{2+2N-NR}$$

and

$$\Phi^{Ey} : clV_{\frac{\varepsilon}{3}}(\theta_1) \times ... \times clV_{\frac{\varepsilon}{3}}(\theta_{NR}) \times F_{\xi^\circ} \times V(\xi^\circ) \to \mathbb{R}^{3+NR}, \text{ for each } y.$$

Given (P_s^r, ζ), assume that, for each y, $\Phi^{Ey} \pitchfork 0$, and that both $\Phi^E \pitchfork 0$, and $\Psi^E \pitchfork 0$. Given that the dimension of the (compact) domains of $\Phi_\xi^{Ey}(.)$, $\Phi_\xi^E(.)$, and $\Psi_\xi(.)$ are smaller than the dimensions of their range, there is an open, dense subset of $V(\xi^\circ)$ such that $\Phi_\xi^{Ey} \pitchfork 0$, $\Phi_\xi^E \pitchfork 0$, and $\Psi_\xi \pitchfork 0$. This means that $\Psi_\xi^{-1}(0) = \emptyset$, and $\Phi_\xi^{Ey-1}(0) = \emptyset$, while either $\Phi_{s\xi}^{E-1}(0) = \emptyset$ or $D_{(\theta^a, \phi)} \Phi_{s\xi}^y(.)$ has full rank at each $(\theta^a, \phi) \in \Phi_\xi^{E-1}(0)$. We postpone the proof that the maps defined above are actually transversal to 0. For the time being, just assume so.

Repeat the procedure for each y and for every $P_s \in$Þ$_s$. Iterate the procedure for each possible collection $\theta_N = \{\theta_1, ..., \theta_N\}$ with the properties discussed above. Finally, repeat it for each $N \in \mathbb{N}$. This procedure gives us a countable collection of open, dense subsets of $V(\xi^\circ)$. Take the intersection of all these subsets and of the set Ξ' (i.e., the set of economies such that, given T, $G^{-1}(0)$ contains a finite number of isolated points), whose existence has been established in the proof of Lemma 1. Define as $V'(\xi^{\circ})$ the non-empty, residual (hence, dense) set so obtained.

Pick $\xi' \in V'(\xi^{\circ})$, and any interior SSE of ξ', $\left(\theta' = G^{-1}(0) \cap (\theta_\ell, \theta_h), \phi\right)$. Given that $\xi' \in \Xi'$, $G^{-1}(0) \cap (\theta_\ell, \theta_h)$ is a finite set, with, say, N elements. Partition the set $G^{-1}(0) \cap (\theta_\ell, \theta_h)$ into two vectors $\left(\theta^{a'}, \theta^{b'}\right)$, such that $\frac{\partial G(\theta,\phi;P_s^r,\zeta,\xi)}{\partial \theta}|_{\theta=\theta_n} = 0$ if and only if $\theta_n \in \theta^{b'}$.

We now show that, by construction, $\left(\theta', \phi'\right) \in clV_\varepsilon(\theta_1) \times ... \times clV_\varepsilon(\theta_N) \times F_{\xi^{\circ}}$, for some vector $\bar{\theta}$, with N rational coordinates and some (P_s^{r*}, ζ^*). Indeed, let

$$\varepsilon' = \min\left\{\min_{n,n'} dist\left(\theta'_n, \theta'_{n'}\right), \min_n dist\left(\theta'_n, \theta_h\right), \min_n \left(\theta_n, \theta'_\ell\right)\right\} > 0.$$

Pick any sequence of N elements with rational coordinates $\{\theta^\nu \equiv (\theta_1^v, ..., \theta_N^v)\}_{v=1}^\infty$ such that $\theta^\nu \to \theta'$ (which exists, by definition of $\mathbb{R}$). Let $\{\varepsilon^\nu\}_{v=1}^\infty$ be the associated sequence of values of ε (as constructed above). Evidently, $\varepsilon^\nu \to \varepsilon'$. Given that any neighborhood $V_{\frac{\varepsilon'}{2}}\left(\theta^{'}\right)$ contains a vector with rational coordinates, for v sufficiently large,

$$\left(\theta^{'}, \phi'\right) \in clV_\varepsilon(\theta_1) \times ... \times clV_\varepsilon(\theta_N) \times F_{\xi^{\circ}}.$$

Hence, for some (P_s^r, ζ), $\Phi^E\left(\theta^{a'}, \phi'; P_s^r, \zeta, \xi'\right) = 0$. By construction and TT, given $\xi' \in V'(\xi^{\circ})$, this implies that $rankD_{(\theta^a,\phi)}\Phi_s^E\left(\theta^a, \phi; P_s^{r*}, \zeta^*, \xi'\right) = (2 + NR)$ at $\left(\theta^{a\prime}, \phi'\right)$. Transversality also implies that $\Psi\left(\theta^{'}, \phi'; P_s^{r*}, \zeta^*, \xi\right) = 0$ has no solutions, so that $\theta^{a'} = \theta^{'}$. Finally, given that $\Phi^{Ey}\left(\theta^{a'}, \phi'; P_s^{r*}, \zeta^*, \xi\right) = 0$ has no solution, $G(\theta_y, \phi; \xi) \neq 0$, for $y = \ell, h$. Hence, the Thm. holds at each interior SE of such a ξ', and, therefore, all the interior SSE of ξ' are regular, i.e., ξ' is a regular economy. In turn, regularity of SSE implies that there is some small open neighborhood of economies $V\left(\xi'\right)$ such that, for each $\xi \in V\left(\xi'\right)$, regular SSE are also described by the same collection of smooth functions. Continuity implies that, for $V\left(\xi'\right)$ sufficiently small, each $\xi \in V\left(\xi'\right)$ has only regular SSE (otherwise, we could construct a sequence of non regular equilibria converging to a regular SSE of ξ'. This is impossible). This establishes the Thm. for an open, dense subset of Ξ_T.

We still have to establish the key fact, i.e., that the maps defined above are actually transversal to 0. As we have done several times already, we compute derivatives with respect to vacancy costs, v, and the function $c(\theta)$. Define

$$c(\theta, d^c) = c(\theta) + \sum_{n=1}^{N} \varphi_n(\theta_n)\left(d_{0n}^c + d_{1n}^c \theta\right) + \varphi_\ell(\theta_\ell)\, d_{0\ell}^c + \varphi_h(\theta_h)\, d_{0h}^c$$

where the bump functions $\varphi_n(\theta_n)$ are positive only on the non-intersecting neighborhoods $V_{2\varepsilon}(\theta_n)$. Then, for each given (P_s^{r*}, ζ^*)

$$D_{(v,d)}\Phi^E(\theta^a, \phi; \xi) = \begin{bmatrix} dv & d_{01}^c & \cdots & d_{0NR}^c \\ D_v\Phi^{'}(.) & 0 & 0 & 0 \\ 0 & \left(1 - e^{rT}\right) & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(1 - e^{rT}\right) \end{bmatrix},$$

$D_{(v,d)}\Psi\left(\theta,\phi;\xi\right)=$

$$\left[\begin{array}{ccccccc} dv & d^c_{0NR+1} & \cdots & d^c_{0N} & d^c_{1NR+1} & \cdots & d^c_{1N} \\ D_{(v,d)}\Phi^E_s & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \left(1-e^{rT}\right) & \vdots & 0 & \left(1-e^{rT}\right)\theta & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \left(1-e^{rT}\right) & 0 & \vdots & \left(1-e^{rT}\right)\theta \\ \vdots & \vdots & \vdots & 0 & \left(1-e^{rT}\right) & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & \left(1-e^{rT}\right) \end{array}\right],$$

and

$$D_{(v,d)}\Phi^{Ey}\left(\theta,\phi;\xi\right)=\left[\begin{array}{cc} dv & d^c_{0y} \\ D_{(v,d)}\Phi^E_s & 0 \\ 0 & \left(1-e^{rT}\right) \end{array}\right], \quad \text{for} \quad y=\ell,h.$$

Given that $\text{rank} D_v\Phi'\left(\theta^a,\phi;P^{r*}_s,\zeta^*,\xi\right)=2$, all the matrices above have full rank. Hence, all the maps are transversal to 0. ■

8.4. Appendix 4: Proof of Theorem 3

Proof of Theorem 4. In this Appendix, to avoid possible misunderstandings, we will denote $\Omega^{0e},\Omega^{0e\alpha}$ etc. the sets chosen by the planner. Also, we are still replacing $\Phi_{\Omega^{0e}}\left(\phi;\xi\right)$ and $\Phi\left(\phi;\xi\right)$ with $\Phi'_{\Omega^{0e}}\left(\phi;\xi\right)$ and $\Phi'\left(\phi;\xi\right)$, defining now the maps $F^k\left(\Omega^{k\alpha}\right)$ in the obvious way.

Consider the set Ξ" of the economies such that every regular interior SSE allocation, if it exists, is not WCPO. As above, we proceed by contradiction showing first that Ξ" is dense in Ξ. Assume that there is some open set $V\left(\xi^\circ\right)\subset\Xi\backslash\Xi$". Without loss of generality (in view of Thm. 3), we can assume that each $\xi\in V\left(\xi^\circ\right)$ is a regular economy with SSE described by a collection of smooth functions $\left(\phi^1\left(\xi\right),...,\phi^N\left(\xi\right)\right)$. Consider the SSE described by $\phi^1\left(\xi\right)$. Regularity also implies that, for each $\xi\in V\left(\xi^\circ\right)$, the correspondence $G^{-1}\left(0\right)$ evaluated at $\left(\phi^1\left(\xi\right),\xi\right)$ is described by a finite collection of smooth functions $\left(\theta_1\left(\phi^1\left(\xi\right),\xi\right),...,\theta_M\left(\phi^1\left(\xi\right),\xi\right)\right)$. Moreover, at each SSE $G(\theta_\ell,\phi^1\left(\xi\right),\xi)\neq0$ and $G(\theta_h,\phi^1\left(\xi\right),\xi)\neq0$.

To avoid unnecessary problems, it is convenient to restrict further the optimization problem of the planner, by requiring that

1. the set Ω^{0e} has the same structure of the set $\Theta^{e\alpha}$ associated with the SSE,
2. the interior boundary points $\{\theta_1,...,\theta_M\}$ lie in some small non-intersecting open neighborhoods of the SSE boundary points.

For instance, if $\Theta^{e\alpha}=[\theta_\ell,\theta^*_1]\cup[\theta^*_M,\theta_h]\cup_2^{M-1}\left[\theta^*_m,\theta^*_{m+1}\right]$, the (modified) planner's optimization problem is

$$\begin{array}{rcl} (\theta_1,...,\theta_M,\phi) & \in & \arg\max\ P(\theta_1,...,\theta_M,\phi)\ \ \textit{subject to}\ \ \Phi_{\Omega^{0e}}\left(\phi;\xi\right)=0 \\ \theta^*_m-\varepsilon & < & \theta_m<\theta^*_m+\varepsilon,\ \text{for each } m. \end{array} \quad (11)$$

Evidently, (11) may not have a solution (because the constraint is not compact). However, *if* the SSE is WCPO, the SSE vector $(\theta_1^*, ..., \theta_M^*, \phi^*)$ is also a solution to the stated optimization problem. Our approach is to show that the necessary FOCs of (11) are typically violated at a SSE. This immediately implies that the SSE is not a solution to (11) and, *a fortiori*, that it is not WCPO.

The FOCs for an interior solution to the Lagrange problem (11), $max\Lambda(\theta_1, ..., \theta_m, \phi, \delta)$, are given by

$$
\begin{array}{ll}
i. & \frac{\partial \Lambda}{\partial \phi^k} = \frac{\partial P}{\partial \phi^k} - \delta^k \frac{\partial \Phi^{k\prime}_{\Omega^{0e}}(\phi;\xi)}{\partial \phi^k} = 0, \text{ each } k, \\
ii. & \frac{\partial \Lambda}{\partial \theta_m} = \frac{\partial P}{\partial \theta_m} - \sum_k \delta^k \frac{\partial \Phi^{k\prime}_{\Omega^{0e}}(\phi;\xi)}{\partial \theta_m} = 0, \text{ each } m, \\
iii. & \frac{\partial \Lambda}{\partial \delta} = -\Phi'_{\Omega^{0e}}(\phi;\xi) = 0,
\end{array}
$$

where $(\delta^k, \delta^m) \in \mathbb{R}^2_+$ is the vector of Lagrange multipliers.

The complete system of equations defining a SSE is given by

$$
\begin{array}{ll}
a. & \Phi'(\phi;\xi) = 0, \\
b. & G(\theta_m, \phi; \xi) = 0, \text{ each } m.
\end{array}
$$

We now show that, for a generic set of parameters ξ, if $(\theta_1, ..., \theta_M, \phi)$ solves $(a - b)$, there is no strictly positive vector of Lagrange multipliers such that it also solves $(i - iii)$.

Define the transformation $\Psi(\theta, \phi, \delta; \xi) : \mathbb{R}^{M+4}_{++} \times V(\xi^\circ) \to \mathbb{R}^{2M+4}$, given by

$$
\Psi(\theta, \phi, \delta; \xi) = \begin{bmatrix} \frac{\partial \Lambda}{\partial \phi} \\ \Phi'_{\Omega^{0e}}(\phi;\xi) \\ \frac{\partial \Lambda}{\partial \theta_m}, \quad \text{each } m \\ G(\theta_m, \phi; \xi), \quad \text{each } m \end{bmatrix}.
$$

Evidently, a WCPO allocation must satisfy $\Psi(\theta, \phi, \delta; \xi^\circ) = 0$. Indeed, the first three blocks of $(M+4)$ equations are the FOCs a WCPO allocation must satisfy. The last M equations must be satisfied for $(\theta_1, ..., \theta_1)$ to be the set of (local) threshold values at the SSE. Assume that $\Psi(\theta, \phi, \delta; \xi^\circ) \pitchfork 0$. Then, for each ξ in some dense subset of $V(\xi^\circ)$, $\Psi_\xi \pitchfork 0$, which implies that $\Psi_\xi^{-1}(0) = \emptyset$. Hence, our proof reduces to show that $\Psi(\theta, \phi, \delta; \xi^\circ) \pitchfork 0$. By direct computation, $D_{(\delta,\xi)}\Psi(\theta, \phi, \delta; \xi)$ contains the following submatrix

$$
\begin{bmatrix}
d\delta^e & d\delta^{ne} & dv^e & dv^{ne} & dc & d\varphi \\
-\frac{\partial \Phi^{e\prime}_{\Omega 0e}}{\partial \phi^e} & 0 & * & 0 & 0 & 0 \\
0 & -\frac{\partial \Phi^{ne\prime}_{\Omega 0e}}{\partial \phi^{ne}} & 0 & * & 0 & 0 \\
0 & 0 & \frac{\partial \Phi^{e\prime}_{\Omega 0e}}{\partial v^e} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial \Phi^{ne\prime}_{\Omega 0e}}{\partial v^{ne}} & 0 & 0 \\
* & * & * & * & \begin{bmatrix} \frac{\partial^2 \Lambda}{\partial \theta \partial dc} \end{bmatrix} & \begin{bmatrix} \frac{\partial^2 \Lambda}{\partial \theta \partial d\varphi} \end{bmatrix} \\
0 & 0 & 0 & 0 & \begin{bmatrix} \frac{\partial G^M}{\partial dc} \end{bmatrix} & \begin{bmatrix} \frac{\partial G^M}{\partial d\varphi} \end{bmatrix}
\end{bmatrix},
$$

where "*" denotes (possibly) non-zero coefficients, while the first row reports the variable we are differentiating with respect to. The meaning of $(d\delta, dv)$ is clear. The associated columns are linearly independent because $\frac{\partial \Phi^{k\prime}_{\Omega 0e}}{\partial \phi^k} < 0$ and $\frac{\partial \Phi^{k\prime}_{\Omega 0e}}{\partial v^k} < 0$, each k. The last two variables denote derivatives with respect to parameters affecting the functions $(c(\theta), f^k(\theta), b^k(\theta))$. Pick a collection of M open balls of radius ε centered on $\theta_m, V_\varepsilon(\theta_m)$, such that $V_{2\varepsilon}(\theta_m) \cap V_{2\varepsilon}(\theta_{m'}) \neq \emptyset$, for each pair θ_m and

$\theta_{m'}$. Also, pick a collection of smooth bump functions $\psi_m(\theta)$, such that $\psi_m(\theta) = 1$ for $\theta \in V_\varepsilon(\theta_m)$, $\psi_m(\theta) = 0$ for $\theta \in V_{2\varepsilon}(\theta_m)$. Define

$$c(\theta; d_c) = c(\theta) + \sum_m \psi_m(\theta)\, d_{c_m},$$

$$f^e(\theta; d_\varphi) = f^e(\theta) + \sum_m \psi_m(\theta)\, d_{\varphi_m},$$

and

$$b^e(\theta; d_\varphi) = b^e(\theta) + \sum_m \psi_m(\theta)\, d_{\varphi_m}.$$

Bear in mind that these perturbations have no direct effect on the functions $F^k(.)$, each k. Let $G^M(.) = [G(\theta_1, .), ..., G(\theta_M, .)]$. Clearly, $D_{dc} G^M = \left(1 - e^{\gamma T}\right)[I]$, where $[I]$ is the $M \times M$ identity matrix. Given that

$$\begin{aligned}\frac{\partial \Lambda}{\partial \theta_m} &= (-1)^{\chi(\theta_m)} e^{-\gamma T} \left[T(1, \theta_m, \phi; \xi) - \left(\frac{\alpha(\theta_m)\gamma v^e \phi^e}{\gamma + \pi^e(\phi^e)} + \left(1 - e^{\gamma T} - \alpha(\theta_m)\right) \frac{\gamma v^{ne} \phi^{ne}}{\gamma + \pi^{ne}(\phi^{ne})} \right) \right] \\ &\quad - \sum_k \delta^k \frac{\partial \Phi^{k\prime}_{\Omega^{0e}}(\phi; \xi)}{\partial \theta_1},\end{aligned}$$

we have

$$\left[\frac{\partial^2 \Lambda}{\partial \theta \partial dc}\right] = e^{-\gamma T}\left(1 - e^{\gamma T}\right) \begin{bmatrix} (-1)^{\chi(\theta_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^{\chi(\theta_M)} \end{bmatrix}.$$

Given that $\int_{\Theta^{0e}} c(\theta)\, d\theta$ does not directly affect the first four rows of $\Psi_{\theta_1}(\phi, \delta; \xi)$, the structure of column dc follows immediately. Consider now the last column. By direct computation,

$$D_{d_\varphi} G^M = \frac{\beta\pi(\phi^e) - \gamma}{\gamma + \beta\pi(\phi^e)} \begin{bmatrix} \alpha(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha(\theta_M) \end{bmatrix}.$$

Given that $\frac{\partial F^k(\Omega^{k\alpha})}{\partial d_\varphi} = \frac{\partial A^k(\phi^k)}{\partial d_\varphi} = 0$, $\frac{\partial \Phi^{e\prime}_{\Omega^0}}{\partial d_\varphi} = \frac{\partial \Phi^{ne\prime}_{\Omega^0}}{\partial d_\varphi} = 0$. Given that

$$\begin{aligned}\frac{\partial \Lambda}{\partial \phi^e} &= \frac{\gamma e^{-\gamma T}}{(\gamma + \pi^e(\phi^e))^2} \int_{\Theta^{0e}} \alpha(\vartheta) \left(\left(f^e(\theta) - b^e(\theta)\right) \frac{\partial \pi^e}{\partial \phi^e} + \left(\phi^e \frac{\partial \pi^e}{\partial \phi^e} - \gamma - \pi^e(\phi^e) \right) v^e \right) d\vartheta \\ &\quad - \delta^e \frac{\partial \Phi^{e\prime}_{\Omega^{0e}}(\phi; \xi)}{\partial \phi^e},\end{aligned}$$

$\frac{\partial^2 \Lambda}{\partial \phi^e \partial d_\varphi} = 0$, by construction. Evidently, $\frac{\partial^2 \Lambda}{\partial \phi^{ne} \partial d_\varphi} = 0$, because $\frac{\partial \Lambda}{\partial \phi^{ne}}$ does not depend upon (f^e, b^e, c). Finally, by direct computation,

$$\left[\frac{\partial^2 \Lambda}{\partial \theta \partial d_\varphi}\right] = e^{-\gamma T} \frac{\pi(\phi^e) - \gamma}{\gamma + \pi(\phi^e)} \begin{bmatrix} (-1)^{\chi(\theta_1)}\alpha(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^{\chi(\theta_M)}\alpha(\theta_M) \end{bmatrix}.$$

Given the structure of $D_{(\delta,\xi)}\Psi(\theta,\phi,\delta;\xi)$, to prove that $\Psi \pitchfork 0$, it suffices to show that the bottom right $2M \times 2M$ matrix has full rank. Divide the first M columns by $\left(1-e^{\gamma T}\right)$, column $(M+1)$ by $\alpha(\theta_1)$, column $(M+2)$ by $\alpha(\theta_2)$. Finally, divide its first row by $e^{-\gamma T}(-1)^{\chi(\theta_1)}$, the second row by $e^{-\gamma T}(-1)^{\chi(\theta_2)}$ and so on (up to row M). The matrix is now reduced to

$$\begin{bmatrix} I & \frac{\pi(\phi^e)-\gamma}{\gamma+\pi(\phi^e)}[I] \\ I & \frac{\beta\pi(\phi^e)-\gamma}{\gamma+\beta\pi(\phi^e)}[I] \end{bmatrix}.$$

Given that $\beta < 1$, this matrix has obviously full rank $2M$. Hence, $\Psi \pitchfork 0$ and, therefore, by TT, there is a dense subset $D^1 \subset D$ such that $\Psi_d^{-1}(0) = \emptyset$. Given that $\Psi_\xi(\theta,\phi,\delta) : \mathbb{R}^{M+4}_{++} \to \mathbb{R}^{2M+4}$, for each economy in this dense set, $\Psi_\xi(\theta,\phi,\delta) = 0$ has no solution. Regularity of the SSE immediately implies that D^1 is also open. Given that the number of equilibria of $\xi \in V(\xi^\circ)$ is finite, we can iterate the same procedure, obtaining a finite collection of open, dense subsets of D, call them $D^1, ..., D^N$. Their intersection is also an open, dense subset of D. Hence, all the regular SSE of economies in this set are not WCPO. This contradicts our initial claim and, therefore, Ξ" is dense. Regularity of SSE immediately implies that Ξ" is also open. ■

8.5. Appendix 5: Two special cases

Proof of Lemma 4. By direct computation,

$$\begin{aligned}\frac{\partial G}{\partial \theta} &= \frac{\partial \alpha}{\partial \theta}\left[\frac{\pi(\phi^e)\beta f^e(\theta)}{r+\beta\pi(\phi^e)} - \frac{\pi(\phi^{ne})\beta f^{ne}(\theta)}{r+\beta\pi(\phi^{ne})}\right] \\ &\quad + \alpha(\theta)\frac{\pi(\phi^e)\beta}{r+\beta\pi(\phi^e)}\frac{\partial f^e(\theta)}{\partial\theta} + \left(1-e^{rT}-\alpha(\theta)\right)\frac{\pi(\phi^{ne})\beta}{r+\beta\pi(\phi^{ne})}\frac{\partial f^{ne}(\theta)}{\partial\theta}.\end{aligned}$$

Multiplying by θ and rearranging, we get

$$\begin{aligned}\theta\frac{\partial G}{\partial \theta} &= \eta_\alpha(\theta)\alpha(\theta)\left[\frac{\pi(\phi^e)\beta f^e(\theta)}{r+\beta\pi(\phi^e)} - \frac{\pi(\phi^{ne})\beta f^{ne}(\theta)}{r+\beta\pi(\phi^{ne})}\right] + \\ &\quad \eta_{f^e}(\theta)\left[\alpha(\theta)\frac{\pi(\phi^e)\beta f^e(\theta)}{r+\beta\pi(\phi^e)} + (1-e^{rT}-\alpha(\theta))\frac{\pi(\phi^{ne})\beta f^{ne}(\theta)}{r+\beta\pi(\phi^{ne})}\frac{\eta_{f^{ne}}(\theta)}{\eta_{f^e}(\theta)}\right].\end{aligned}$$

If $\eta_\alpha(\theta) \geq 0$ the first term is positive. When $\frac{\eta_{f^{ne}}(\theta)}{\eta_{f^e}(\theta)} = 1$, the second term in square brackets is $G(\theta,\phi)$ and it must be nil at each $\theta \in G^{-1}(0)$. Hence, this term is positive as long as $\frac{\eta_{f^{ne}}(\theta)}{\eta_{f^e}(\theta)} < 1$. Therefore, under the assumptions of case a, $\frac{\partial G}{\partial\theta} > 0$ at each $\theta \in G^{-1}(0)$. This establishes the first claim. The second is similarly proved. ■

Proof of Proposition 4. Remember that the maintained assumptions are $\det D_\phi \Phi'(.) > 0$ and $\frac{\partial F^{ne}}{\partial\theta} > 0$. In the case of complementarity, $\frac{\partial G}{\partial\theta} > 0$ at $\theta \in G^{-1}(0)$. Hence, by applying the IFT to the map $G(\theta,\phi;\xi)$, at $\theta^* \in G^{-1}(0)$,

$$\begin{bmatrix} \frac{\partial\theta^*}{\partial\phi^e} & \frac{\partial\theta^*}{\partial\phi^{ne}} & \frac{\partial\theta^*}{\partial f^e} & \frac{\partial\theta^*}{\partial f^{ne}} & \frac{\partial\theta^*}{\partial c} & \frac{\partial\theta^*}{\partial\alpha} & \frac{\partial\theta^*}{\partial b^e} & \frac{\partial\theta^*}{\partial b^{ne}} & \frac{\partial\theta^*}{\partial v^k} & \frac{\partial\theta^*}{\partial\pi^e} & \frac{\partial\theta^*}{\partial\pi^{ne}} \\ - & + & - & + & + & - & - & + & 0 & + & - \end{bmatrix}$$

In the sequel, we just present the explicit values of the vectors $\nabla_\xi\phi$, as computed applying the IFT to the map $\Phi(\phi;\xi)$.

$$
\left[\begin{array}{ccc}
 & \nabla_\xi \phi^e & \nabla_\xi \phi^{ne} \\
\boldsymbol{\nabla}_{f^e} & \frac{F^e(.)\left[\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^{ne}}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\right]-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial f^e}}{-\det D_\phi\Phi} & \frac{-\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^e}F^e(.)-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial f^e}}{-\det D_\phi\Phi} \\
\boldsymbol{\nabla}_{f^{ne}} & \frac{-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial f^{ne}}-\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^{ne}}F^{ne}(.)}{-\det D_\phi\Phi} & \frac{\left[\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^e}-\frac{\partial A^e}{\partial\phi^e}\right]F^{ne}(.)-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial f^{ne}}}{-\det D_\phi\Phi} \\
\boldsymbol{\nabla}_c & \frac{-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial c}}{-\det D_\phi\Phi} & \frac{-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial c}}{-\det D_\phi\Phi} \\
\boldsymbol{\nabla}_\alpha & \frac{-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial\alpha}}{-\det D_\phi\Phi} & \frac{-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\alpha}}{-\det D_\phi\Phi} \\
\nabla_{b^e} & \frac{-\left[\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^{ne}}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\right]-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial b^e}}{-\det D_\phi\Phi} & \frac{\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^e}-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial b^e}}{-\det D_\phi\Phi} \\
\nabla_{b^{ne}} & \frac{\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^{ne}}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta}\frac{\partial\theta^*}{\partial b^{ne}}}{-\det D_\phi\Phi} & \frac{-\left[\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^e}-\frac{\partial A^e}{\partial\phi^e}\right]-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial b^{ne}}}{-\det D_\phi\Phi} \\
\nabla_{v^e} & \frac{-\left[\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^{ne}}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\right]\frac{r+\beta\pi(\phi^e)}{(1-\beta)q(\phi^e)}}{-\det D_\phi\Phi} & \frac{\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^e}\frac{r+\beta\pi(\phi^{ne})}{(1-\beta)q(\phi^{ne})}}{-\det D_\phi\Phi} \\
\nabla_{v^{ne}} & \frac{\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta^*}{\partial\phi^{ne}}\frac{r+\beta\pi(\phi^{ne})}{(1-\beta)q(\phi^{ne})}}{-\det D_\phi\Phi} & \frac{-\left[\frac{\partial F^e}{\partial\theta}\frac{\partial\theta}{\partial\phi^e}-\frac{\partial A^e}{\partial\phi^e}\right]\frac{r+\beta\pi(\phi^{ne})}{(1-\beta)q(\phi^{ne})}}{-\det D_\phi\Phi} \\
\nabla_{q^e} & \frac{-\left[\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta}{\partial\phi^{ne}}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\right]\frac{\partial A^e}{\partial q^e}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta}{\partial q^e}}{-\det D_\phi\Phi} & \frac{\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta}{\partial\phi^e}\frac{\partial A^{ne}}{\partial q^e}-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta}{\partial q^e}}{-\det D_\phi\Phi} \\
\nabla_{q^{ne}} & \frac{\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta}{\partial\phi^{ne}}\frac{\partial A^{ne}}{\partial q^{ne}}-\frac{\partial A^{ne}}{\partial\phi^{ne}}\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta}{\partial q^{ne}}}{-\det D_\phi\Phi} & \frac{-\left[\frac{\partial F^e}{\partial\theta^*}\frac{\partial\theta}{\partial\phi^e}-\frac{\partial A^e}{\partial\phi^e}\right]\frac{\partial A^{ne}}{\partial q^{ne}}-\frac{\partial A^e}{\partial\phi^e}\frac{\partial F^{ne}}{\partial\theta^*}\frac{\partial\theta}{\partial q^{ne}}}{-\det D_\phi\Phi}
\end{array}\right].
$$

■

Proof of Proposition 5. Now, we have

$$
\left[\begin{array}{ccccccccccc}
\frac{\partial\theta^*}{\partial\phi^e} & \frac{\partial\theta^*}{\partial\phi^{ne}} & \frac{\partial\theta^*}{\partial f^e} & \frac{\partial\theta^*}{\partial f^{ne}} & \frac{\partial\theta^*}{\partial c} & \frac{\partial\theta^*}{\partial\alpha} & \frac{\partial\theta^*}{\partial b^e} & \frac{\partial\theta^*}{\partial b^{ne}} & \frac{\partial\theta^*}{\partial v^k} & \frac{\partial\theta^*}{\partial\pi^e} & \frac{\partial\theta^*}{\partial\pi^{ne}} \\
+ & - & + & - & + & + & + & - & 0 & - & +
\end{array}\right].
$$

The result follows immediately. ■

REFERENCES

[1] Acemoglu, D.T., 1996, A microfoundation for social increasing returns in human capital accumulation, *Q.J.E,* 111, 779-804.

[2] Al-Najjar, N.I., 2004, Aggregation and the law of large numbers in large economies, *Games Econ. Behavior,* 47, 1-35.

[3] Alòs-Ferrer, C., 1999, Dynamical Systems with a Continuum of Randomly Matched Agents, *J. Econ. Theory,* 86, 245-267.

[4] Becker, G.S., 1964, *Human capital: A theoretical and empirical analysis with special reference to education*, Columbia University Press, New York.

[5] Becker, S. O., 2006, Introducing time-to-educate in a job search model, *Bulletin of Economic Research,* 58, 61-72.

[6] Ben-Porath, Y., 1967, The production of human capital and the life-cycle of earnings," *J. Pol. Econ.*, 75, 352-365.

[7] Booth, A.L., and M. Coles, 2007, A microfoundation for increasing returns in human capital accumulation and the under-participation trap, *Europ. Econ. Review*, 51, 1661-1681.

[8] Booth, A.L., and M. Coles, 2005, Increasing returns to education and the skills under-investment trap, IZA DP No. 1657.

[9] Booth, A.L., M. Coles, and X. Gong, 2006, Increasing returns to education: theory and evidence, The Australian National University, Centre for Economic Policy Research, D.P. 522.

[10] Burdett, K. and E. Smith, 1996, Education and matching externalities, in Booth, A.L., and D.J. Snower (eds.), *Acquiring Skills: Market Failures, Their Symptoms and Policy Responses,* Cambridge University Press, Cambridge, 65-80.

[11] Burdett, K. and E. Smith, 2002, The low skill trap, *Europ. Econ. Review,* 46, 1439-1451.

[12] Carneiro, P., J.J.Heckman, and E. Vytlacil, 2001, Estimating the return to education when it varies among individuals, mimeo.

[13] Cahuc, P., F. Postel-Vinay, and J.-M. Robin, 2006, Wage bargaining with on-the-job search: Theory and evidence, *Econometrica*, 74, 323-364.

[14] Charlot, O. and B. Decreuse, 2005, Self-selection in education with matching frictions. *Labour Econ.*,12, 251-267.

[15] Charlot, O. and B. Decreuse, 2006, Over-education for the rich vs. under-education for the poor: A search-theoretic microfoundation, GRECAM, Document de Travail 2006-2.

[16] Charlot, O., B. Decreuse, and P. Granier, 2005, Adaptability, productivity, and educational incentives in a matching model, *Europ. Econ. Review*, 49, 1007-1032.

[17] Cunha, F., and J.J. Heckman, 2006, Identifying and estimating the distributions of ex post and ex ante returns to schooling: A survey of recent developments, mimeo.

[18] Cunha, F., J.J. Heckman, and S. Navarro, 2005, Separating uncertainty from heterogeneity in life cycle earnings, The 2004 Hicks Lecture, *Oxford Econ. Papers*, 57, 1-72.

[19] De la Fuente, A., 2003. Human capital in a global and knowledge-based economy. Part II: Assessment at the EU country level. Final Report, European Commission, Directorate General for Employment and Social Affairs.

[20] Duffie, D., and Y. Sun, 2007, Existence of independent random matching, *The Annals of Applied Probability*, 17, 386–419.

[21] Feldman, M., and C. Gilles, 1985, An expository note on individual risk without aggregate uncertainty, *J. Econ. Theory,* 35, 26-32.

[22] Flinn, C.J., and J. Mabli, 2008, Minimum wage and labor market outcomes under search with bargaining, IRP Discussion Paper 1337-08.

[23] Geanakoplos, J., and H. Polemarchakis, 1986, Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete, in W.P. Heller, R.M. Starr and D. Starrett (eds.), *Uncertainty, information and communication: Essays in honor of K.J. Arrow, Vol. III*, Cambridge University Press, Cambridge.

[24] Hirsch, M.W., 1976, *Differential Topology*, Springer-Verlag, Berlin.

[25] Hosios, A.J., 1990, On the efficiency of matching and related models of search and unemployment, *R. Econ. Studies*, 57, 279-298.

[26] Judd, K.J., 1985, The law of large numbers with a continuum of i.i.d. random variables, *J. Econ. Theory*, 35, 19-25.

[27] Laing, D., T. Palivos, and P. Wang, 1995, Learning, matching and growth in a search model, *R. Econ. Studies*, 62.

[28] Mendolicchio, C., D. Paolini, and T. Pietra, 2008, Human capital policies in a two-sector, static economy with imperfect markets, CRENoS Working Paper 2008-08.

[29] OECD, 2004, *Education at a Glance*, OECD, Paris.

[30] Petrongolo, B., C.A. Pissarides, 2001, Looking into the black box: A survey of the matching function, *J. Econ. Lit.*, 39, 390-431.

[31] Roy, A., 1951, Some thoughts on the distribution of earnings, *Oxford Econ. Papers*, 3, 135-146.

[32] Sun, Y., 2006, The exact law of large numbers via Fubini extension and characterization of insurable risks, *J. Econ. Theory*, 126, 31-69.

[33] Tawara, N., 2007, External returns to time-consuming schooling in a frictional labor market, mimeo, University of Chicago.

[34] Willis, R., 1986, Wage determinants: A survey and reinterpretation of human capital earnings functions, in O. Ashenfelter, and R. Layard (eds.), *Handbook of Labor Economics*, Amsterdam, North-Holland.

[35] Willis, R., and S. Rosen, 1979, Education and self-selection, *J. Pol. Econ.*, 87, S7-36.

[36] Yashiv, E., 2003, Bargaining. the value of unemployment, and the behavior of aggregate wages, mimeo.

[37] Yashiv, E., 2006, Evaluating the performance of the search and matching model, *Europ. Econ. Review*, 50, 909–936.

Ultimi Contributi di Ricerca CRENoS

I Paper sono disponibili in: http://www.crenos.it

08/08 *Stefano Usai,* "Innovative Performance of Oecd Regions"
08/07 *Concetta Mendolicchio, Tito Pietra, Dimitri Paolini,* "Human Capital Policies in a Static, Two-Sector Economy with Imperfect Markets"
08/06 *Vania Statzu, Elisabetta Strazzera,* "A panel data analysis of electric consumptions in the residential sector"
08/05 *Marco Pitzalis, Isabella Sulis, Mariano Porcu,* "Differences of Cultural Capital among Students in Transition to University. Some First Survey Evidences"
08/04 *Isabella Sulis, Mariano Porcu,* "Assessing the Effectiveness of a Stochastic Regression Imputation Method for Ordered Categorical Data"
08/03 *Manuele Bicego, Enrico Grosso, Edoardo Otranto,* "Recognizing and Forecasting the Sign of Financial Local Trends Using Hidden Markov Models"
08/02 *Juan de Dios Tena, Edoardo Otranto,* "A Realistic Model for Official Interest Rates Movements and their Consequences"
08/01 *Edoardo Otranto,* "Clustering Heteroskedastic Time Series by Model-Based Procedures"
07/16 *Sergio Lodde,* "Specialization and Concentration of the Manufacturing Industry in the Italian Local Labor Systems"
07/15 *Giovanni Sulis,* "Gender Wage Differentials in Italy: a Structural Estimation Approach"
07/14 *Fabrizio Adriani, Luca G. Deidda, Silvia Sonderegger,* "Over-Signaling Vs Underpricing: the Role of Financial Intermediaries In Initial Public Offerings"
07/13 *Giovanni Sulis,* "What Can Monopsony Explain of the Gender Wage Differential In Italy?"
07/12 *Gerardo Marletto,* "Crossing the Alps: Three Transport Policy Options"
07/11 *Sergio Lodde* "Human Capital And Productivity Growth in the Italian Regional Economies: a Sectoral Analysis"
07/10 *Axel Gautier, Dimitri Paolini,* "Delegation, Externalities and Organizational Design"
07/09 *Rinaldo Brau, Antonello E. Scorcu, Laura Vici,* "Assessing visitor satisfaction with tourism rejuvenation policies: the case of Rimini, Italy"
07/08 *Dimitri Paolini,* "Search and the firm's choice of the optimal labor contract"
07/07 *Giacomo Carboni,* "Shape of U.S. business cycle and long-run effects of recessions"
07/06 *Gregory Colcos, Massimo Del Gatto, Giordano Mion and Gianmarco I.P. Ottaviano,* "Productivity and firm selection: intra-vs international trade"
07/05 *Silvia Balia,* "Reporting expected longevity and smoking: evidence from the share"
07/04 *Raffaele Paci, Stefano Usai,* "Knowledge flows across European regions"
07/03 *Massimo Del Gatto, Giordano Mion and Gianmarco I.P. Ottaviano,* "Trade Integration, firm selection and the costs of non-europe"