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Asymptotic profile and Morse index of nodal radial solutions to the Hénon problem

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Abstract	We compute the Morse index of nodal radial solutions to the Hénon problem $\begin{cases} -\Delta u = x ^{\alpha} u ^{p-1} u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$		
	where <i>B</i> stands for the unit ball in \mathbb{R}^N in dimension $N \ge 3$, $\alpha > 0$ and <i>p</i> is close to the threshold exponent $N + 2 + 2\alpha$		
	for existence of solution	or existence of solutions $p_{\alpha} = \frac{1}{N-2}$, obtaining that either	

$m(u_p) = m$	$\sum_{j=0}^{1+\left\lfloor \alpha/2\right\rfloor}$	N_j	
$m(u_p) = m$	$\sum_{j=0}^{\alpha/2} N_j$	+(m-1)	$N_{1+\alpha/2}$

if α is not an even integer, or

if α is an even number.

Here N_j denotes the multiplicity of the spherical harmonics of order *j*, and *m* stands for the number of nodal zones of *u*. The computation builds on a characterization of the Morse index by means of a one dimensional singular eigenvalue problem, and is carried out by a detailed picture of the asymptotic behavior of both the solution and the singular eigenvalues and eigenfunctions. In particular it is shown that nodal radial solutions have multiple blow-up at the origin, and converge (up to a suitable rescaling) to the bubble shaped solution of a limit problem. As side outcome we see that solutions are nondegenerate for *p* near $P\alpha$, and we give an existence result in perturbed balls.

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Asymptotic profile and Morse index of nodal radial solutions to the Hénon problem

Anna Lisa Amadori¹ · Francesca Gladiali²

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Abstract

² We compute the Morse index of nodal radial solutions to the Hénon problem

$$\begin{cases} -\Delta u = |x|^{\alpha} |u|^{p-1} u & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

where B stands for the unit ball in \mathbb{R}^N in dimension $N \ge 3$, $\alpha > 0$ and p is close to the

threshold exponent for existence of solutions $p_{\alpha} = \frac{N+2+2\alpha}{N-2}$, obtaining that either

$$m(u_p) = m \sum_{j=0}^{1+[\alpha/2]} N_j$$
 if α is not an even integer, or

$$m(u_p) = m \sum_{j=0}^{\alpha/2} N_j + (m-1)N_{1+\alpha/2}$$
 if α is an even number.

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⁸ Here N_j denotes the multiplicity of the spherical harmonics of order j, and m stands for ⁹ the number of nodal zones of u. The computation builds on a characterization of the Morse ¹⁰ index by means of a one dimensional singular eigenvalue problem, and is carried out by a ¹¹ detailed picture of the asymptotic behavior of both the solution and the singular eigenvalues ¹² and eigenfunctions. In particular it is shown that nodal radial solutions have multiple blow-up ¹³ at the origin, and converge (up to a suitable rescaling) to the bubble shaped solution of a limit ¹⁴ problem. As side outcome we see that solutions are nondegenerate for p near p_{α} , and we ¹⁵ give an existence result in perturbed balls.

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16 Contents

17	1	Introduction
18	2	The asymptotic profile of u_p via a "radially extended" version of the Lane–Emden problem
19		2.1 The proof of Propositions 2.2 and 2.3
20		2.2 Some consequences of the convergence result
21	3	The computation of the Morse index
22		3.1 Asymptotics of the singular eigenvalues $\hat{v}_i(p)$ for $i = 1,, m-1$
23		3.2 The last negative eigenvalue
24	4	Nondegeneracy and small perturbations
25	5	Appendix
26	R	eferences

27 1 Introduction

In this paper we continue the project started with [3,5] and use a singular eigenvalue problem to compute the Morse index of nodal radial solutions to semilinear equations. In particular

³⁰ we focus here on the problem

$$\begin{cases} -\Delta u = |x|^{\alpha} |u|^{p-1} u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$
(1.1)

where $\alpha \ge 0$, *B* stands for the unit ball in \mathbb{R}^N in dimension $N \ge 3$, and p > 1. When $\alpha > 0$ problem (1.1) is known as the Hénon problem, since it has been introduced by Hénon in [26] in the study of stellar clusters in radially symmetric settings, in 1973. Later on Ni, in the celebrated paper [30], proved the existence of a critical exponent related with the parameter α , that we denote hereafter by

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$$p_{\alpha} = \frac{N+2+2\alpha}{N-2} \tag{1.2}$$

which gives the threshold between existence and nonexistence of solutions. Using the fact 38 that $H_{0 \text{ rad}}^{1}(B) := \{u \in H_{0}^{1}(B) : u \text{ is radial}\}$ is compactly embedded in $L^{p+1}(B, |x|^{\alpha} dx)$ for 39 every 1 , Ni proved that (1.1) admits a positive radial solution, which is classical.40 The existence of radial solutions can be then extended to the case of nodal solutions with an 41 arbitrary number of zeros (nodes) by means of a procedure introduced in [10] and using again 42 the compactness of the immersion of $H_{0,rad}^1$ into L^{p+1} as for the case of positive solutions. It 43 is also possible to apply a uniqueness result of [31] to have that for any integer $m \ge 1$ there 44 exists only a couple of radial solutions to (1.1) which have exactly m nodal zones, meaning 45 that the set $\{x \in B : u(x) \neq 0\}$ has exactly *m* connected components; they are one the 46 opposite of the other and classical solutions (see, for instance, [5, Proposition 4.1]). 47

⁴⁸ Moreover, a classical Pohozaev argument shows that the Hénon problem (1.1) does not ⁴⁹ admit solutions when it is settled in a smooth bounded domain Ω which is starshaped with ⁵⁰ respect to the origin and $p \ge p_{\alpha}$. Then p_{α} exhibits the same role of the critical exponent ⁵¹ $p^* = \frac{N+2}{N-2}$ for the Lane–Emden problem

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$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$
(1.3)

which corresponds to (1.1) in the case of $\alpha = 0$. As we will see the relations between Hénon and Lane–Emden problems are much deeper. Indeed radial solutions to (1.1) with $\alpha > 0$ can be viewed as radially extended solutions to (1.3) in a sense which will be clarified in Sect. 2.

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s7 which the authors proved that the ground state solutions to (1.1), namely solutions which

⁵⁸ minimizes the Energy functional

$$\mathcal{E}(u) = \frac{1}{2} \int_{B} |\nabla v|^{2} - \frac{1}{p+1} \int_{B} |x|^{\alpha} |u|^{p+1}$$

on the Nehari manifold

$$\mathcal{N} = \left\{ v \in H_0^1(B) : \int_B |\nabla v|^2 = \int_B |x|^{\alpha} |v|^{p+1} \right\}$$

⁶² for $1 are nonradial provided <math>\alpha > 0$ is sufficiently large. Nevertheless ground state ⁶³ solutions to (1.1) maintain a residual symmetry called foliated Schwartz symmetry, which ⁶⁴ appears in other similar contexts in which the symmetry result of Gidas, Ni and Nirenberg ⁶⁵ in [22] does not hold, namely both in the case of annular domains and for nodal solutions.

Let us recall that the Morse index of a solution u to (1.1) is the maximal dimension of a subspace $X \subseteq H_0^1(B)$ where the quadratic form

Author Proof

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 $Q_u(\psi) := \int_B |\nabla \psi|^2 - p|x|^{\alpha} |u|^{p-1} \psi^2 dx$

is negative defined. The quadratic form Q_u is associated with the linearized operator in *B* with Dirichlet boundary conditions

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$$L_u(\psi) := -\Delta \psi - p |x|^{\alpha} |u|^{p-1} \psi.$$

72 Of course the Morse index can be computed counting (with multiplicity) the negative eigen-

values of L_u in $H_0^1(B)$, but also some negative singular eigenvalues. This equivalence and

the characterization of Morse index in terms of the singular eigenvalues of L_u is given in

⁷⁵ details in [3] and will be essential for our aims.

It is well known that ground state solutions have Morse index one since they can be found as minima on the Nehari manifold, which has codimension one. Then the result in [35] says that radial positive solutions to (1.1) can have Morse index greater than 1, when α is large

⁷⁹ enough.

Starting from this consideration, in [2] we computed the Morse index of radial positive 80 solutions to (1.1) showing that it converges to the value 1 + N when $p \rightarrow p_{\alpha}$ and to the 81 value 1 as $p \rightarrow 1$, and we proved a first bifurcation result from the positive solution of the 82 Hénon problem which is, in our opinion, responsible of the symmetry breaking of (1.1). In 83 this last paper a technical assumption, namely that $0 < \alpha \leq 1$, is required to deal with the 84 linearized operator and compute the asymptotic Morse index of radial positive solutions. This 85 assumption is removed here where, taking advantage from the analysis in [3,5] and using 86 a singular eigenvalue problem associated to the linearized operator, the computation of the 87 Morse index is performed for any value of α . Nevertheless the result in [2] puts evidence on 88 the fact that the symmetry breaking phenomenon pointed out in [35] is not related to large 89

values of α , but still holds when $0 < \alpha \leq 1$.

Later it has been proved in [29] that the Morse index of any radial solution to (1.1) goes to ∞ as $\alpha \to \infty$, showing again the symmetry breaking of the ground state solutions, for

 $_{32}$ large values of α . Their result has implications also concerning nodal ground state solutions, for

⁹⁴ namely minima for $\mathcal{E}(u)$ on the nodal Nehari manifold

$$\begin{split} \mathcal{N}_{nod} &= \left\{ v \in H_0^1(B) : v^+ \neq 0, \ \int_B |\nabla v^+|^2 = \int_B |x|^{\alpha} |v^+|^{p+1}, \\ v^- \neq 0, \ \int_B |\nabla v^-|^2 = \int_B |x|^{\alpha} |v^-|^{p+1} \right\} \end{split}$$

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Here s^+ (s^-) stands for the positive (negative) part of *s*. As it is known by [9] that they have Morse index 2, the estimate in [29] implies that the symmetry breaking phenomenon concerns also nodal ground state solutions. A similar consideration appears also in [5] as a consequence of some estimates on Morse index of radial nodal solutions, but only in the case of solutions which change sign.

The fact that the Morse index of any radial solutions to (1.1) diverges as $\alpha \to \infty$ is a clue that the symmetry breaking phenomenon is not related with a nonradial solution whose energy is less than the radial one, but with infinitely many nonradial (nodal) solutions that should arise by bifurcation. Indeed [37] found infinitely many positive multipeak solutions, with arbitrarily large energy, when $p = p^*$, and infinitely many nonradial solutions have been constructed by bifurcation w.r.t. the parameter α in [20] (concerning positive solutions and *p* near p_{α}) and [28] (concerning both positive and nodal solutions and arbitrary p > 1).

In any case the exact Morse index of radial solutions to (1.1), depending on the parameters p and α and on the number of nodal zones m, is still unknown. To the authors' knowledge the only results in this direction are the computations in [5], where a lower bound on the Morse index is presented and it is proved that the radial Morse index is equal to the number of nodal zones, namely the linearized operator L_u has exactly m negative eigenvalues whose related eigenfunction is radial.

Beyond the symmetry breaking the interest of the mathematicians on the Hénon problem 114 (1.1) is due to the richness of its solutions set, which is completely different from the Lane 115 Emden case. For instance [32] produces multipeak solutions in the slightly subcritical range, 116 by the Lyapunov-Schmidt reduction method. Moreover solutions appear also in a critical 117 and supercritical range, namely whether when $p = p^*$ or when $p > p^*$, and of course 118 $p < p_{\alpha}$. Concerning existence of nonradial solutions in the critical case we quote here [34] 119 and the already mentioned [37]. Coming to the supercritical range, [8] produces nonradial 120 positive solutions using minimization in suitable symmetric spaces and [15] produces positive 121 solutions on perturbed balls for generic values of p, by a perturbation argument. Next for all 122 values of the exponent p close to the threshold p_{α} , and any domain containing the origin, 123 we mention [23] concerning existence of positive solutions and also the papers [14] and 124 [13], where nodal bubble tower solutions are constructed by a Lyapunov–Schmidt reduction 125 method when α is not an even integer, respectively for $\alpha > 0$ and $\alpha > -2$. 126

In this paper we want to fill the gap on the exact value of the Morse index of radial solutions to (1.1), and, considering $\alpha \ge 0$ as a fixed parameter we compute the Morse index of any radial solution to (1.1) in a left neighborhood of the critical exponent p_{α} . To state our main result we denote by $\left[\frac{\alpha}{2}\right] = \max\left\{k \in \mathbb{N} : k \le \frac{\alpha}{2}\right\}$ the integer part of $\frac{\alpha}{2}$, and by $N_j = \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}$ the multiplicity of $\lambda_j = j(N + j - 2)$ as an eigenvalue for the Laplace–Beltrami operator on the sphere \mathbb{S}_{N-1} . Moreover, understanding that for $\alpha = 0$ a solution to (1.1) is exactly a solution to (1.3) and $p_{\alpha} = p^*$, we can state:

Theorem 1.1 Let u_p be any radial solution to (1.1) with m nodal zones and let $\alpha \ge 0$. Then there exists $p^* \in (1, p_\alpha)$ such that for any $p \in [p^*, p_\alpha)$ we have either

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(1.4)

138 $as \alpha > 0$ is not an even integer, or

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$$m(u_p) = m \sum_{j=0}^{\frac{\alpha}{2}} N_j + (m-1)N_{1+\frac{\alpha}{2}}.$$
(1.5)

if $\alpha = 0$ or it is an even number.

This result is inspired by some previous papers on the Morse index of nodal radial solutions to the Lane Emden problem (1.3) in dimension $N \ge 3$, see [18] and in dimension N = 2, see [19], and to the possibility to obtain from its knowledge some existence results of nonradial nodal solutions whose nodal set, namely { $x \in B : u(x) = 0$ }, does not touch the boundary of *B*, as in [25]. It is worth noticing that reading formula (1.5) for $\alpha = 0$ we get

$$m(u_p) = m + (m-1)N$$

for *p* close to the critical exponent p^* , which is the exact formula obtained in [18] for solution to (1.3). As far as $\alpha \in (0, 2)$, (1.4) comes into play and the Morse index is larger, precisely $m(u_p) = m(1+N)$, highlighting the fact that the Morse index increases with α and it changes corresponding exactly to the even values of α .

To have a precise idea of the Morse index of u_p we observe that for small values of α these values are:

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$$\begin{array}{ll} \alpha = 0 & m(u_p) = m + (m-1)N \\ 0 < \alpha < 2 & m(u_p) = m + mN \\ \alpha = 2 & m(u_p) = m + mN + (m-1)N_2 \\ 2 < \alpha < 4 & m(u_p) = m + mN + mN_2 \\ \alpha = 4 & m(u_p) = m + mN + mN_2 + (m-1)N_3 \end{array}$$

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and so on, showing that the Morse index corresponding to the integer values of α is different from every other value for nodal solutions, i.e. for $m \ge 2$. This seems to be a new phenomenon. As mentioned before, Theorem 1.1 brings new informations also in simplest case of

positive solutions (m = 1). In that case formulas (1.4) and (1.5) can be written as

$$m(u_p) = \sum_{j=0}^{k} N_j$$
 if $2(k-1) < \alpha \le 2k$ (1.6)

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for *p* near p_{α} . Equation (1.6) extends the computation made in [2] for $1 < \alpha \le 2$ and describes the different values of the limit for larger values of α . As we have already remarked this last estimate was the crucial part for the bifurcation result in [2], since we have already noticed that the Morse index of positive radial solutions converges to 1 as $p \rightarrow 1$. In a similar manner we expect that formulas (1.4) and (1.5) are responsible of a nonradial bifurcation from nodal radial solutions to (1.1), since the Morse index for *p* close to 1 has been computed in [1] obtaining

$$n(u_p) = 1 + \sum_{i=1}^{m-1} \sum_{j=0}^{\lceil J_i - 1 \rceil} N_j, \quad \text{for } J_i = \frac{(2+\alpha)\beta_i - (N-2)}{2},$$

where $\lceil \cdot \rceil$ stands for the ceiling function $\lceil s \rceil = \min\{n \in \mathbb{N} : n \ge s\}$. The parameters β_i appearing here are linked to the zeros of the Bessel functions of first kind

$$\mathcal{J}_{\beta}(r) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k+1+\beta)} \left(\frac{r}{2}\right)^{2k+\beta}, \quad r \ge 0.$$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

More precisely β_i is characterized as the unique positive parameter for which the *i*th zero of \mathcal{J}_{β_i} coincides with the *m*th zero of $\mathcal{J}_{\frac{N-2}{2+\alpha}}$. Even though the values of the zeros of the Bessel functions (and therefore the parameters β_i) can be computed only by numerical approximations, one can see that for nodal solutions the Morse index near p = 1 is greater than the one near $p = p_{\alpha}$. Therefore a change in the Morse index appears at some values of p, and a nonradial bifurcation should arise for every $\alpha \ge 0$.

Lastly we compare formulas (1.4) and (1.5) with the estimate from below of the Morse index obtained in Theorem 1.1 in [5] (see also Theorem 1.3 in the same paper), which holds for any $p \in (1, p_{\alpha})$ and $\alpha \ge 0$ and states

$$m(u_p) \ge 1 + (m-1) \sum_{j=0}^{1+\lfloor \frac{\alpha}{2} \rfloor} N_j.$$
 (1.7)

For positive solutions (m = 1) it is known that this bound is optimal because the Morse index is equal to 1 when the exponent p approaches the value 1. For nodal solutions, in the case of Lane-Emden problem ($\alpha = 0$) in dimension $N \ge 3$, the estimate from below is attained for p near the critical exponent $p^* = \frac{N+2}{N-2}$ (see [18]). This is not the case anymore for the Hénon problem, because the exact value obtained in Theorem 1.1 overpasses the estimate from below.

Let us spend some words on how we prove Theorem 1.1. First we exploit the character-189 ization of the Morse index and the decomposition of some singular eigenvalues established 190 in [3] and we relate the computation of the Morse index of any radial nodal solution, with m 191 nodal zones, to the knowledge of *m* negative singular radial eigenvalues, see Proposition 3.2. 192 Next we study their asymptotic behavior as $p \to p_{\alpha}$ together with the asymptotic profile of 193 the associated eigenfunctions, which is needed to deal with the last negative singular eigen-104 value. This study furnishes immediately Theorem 1.1 as a consequence of Proposition 1.4 195 of [3] and Theorem 1.3 in [5]. It also shows that the bound (1.7) is obtained by estimating 196 in a sharp way the singular radial eigenvalues: actually the first m-1 eigenvalues reach 197 their upper bound for p near p_{α} , giving the minimal contribution to the Morse index. In 198 the Lane-Emden problem the contribution coming from the last eigenvalue is constant and 199 200 therefore it does not influence the asymptotic behavior of the Morse index. On the contrary in the Hénon problem the contribution of the last eigenvalue varies, and it is maximal for p 201 near p_{α} , minimal when p is near 1. It is thus clear that in the case of $\alpha = 0$ the behavior 202 of the m-1 singular negative eigenvalues is sufficient to compute the Morse index, while 203 when $\alpha > 0$ also the last negative eigenvalue comes into play and its estimate is the most 204 difficult one. 205

The description of the asymptotic behavior of the singular eigenvalues and eigenfunctions relies on the asymptotic analysis of the nodal radial solutions to (1.1) with *m* nodal zones, which is indeed the second main aim of this paper. Let us remark that for the Hénon problem the asymptotic profile is known only in the case of positive solutions. Precisely [2] describes the limit of the radial solution when $p \rightarrow p_{\alpha}$ and α is a fixed parameter, while [11] studies the limit of both the radial and the ground state solution as $\alpha \rightarrow \infty$ and *p* is fixed.

Here we are interested in the limit of the nodal radial solution when the exponent p approaches the threshold p_{α} , and to proceed with the further study of the related eigenvalues we need to know the limit problem to which the solution converges and the behavior of its critical points and values. Concerning the Lane–Emden problem (1.3) these topics have been the subject of some interesting papers, [18] and [19] among others. Solutions to (1.3) indeed tend to concentrate in the origin as showed in [33], and admit a limit problem which can be

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182

Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

used, for instance, to construct concentrating solutions in more general domains and with more general nonlinearities. This aspect is different when the dimension is 2 (and $p \rightarrow \infty$) or higher, so the two cases have to be treated separately. The Hénon problem (1.1) shares the same duality: indeed when N = 2 radial solutions exhibit a different limit problem and a different way to concentrate. For this reason we focus here on the case of $N \ge 3$ while we refer to the paper [4], which contains different conclusions, for the study of the asymptotic behavior of u_p and of its Morse index in the case of N = 2.

To state the related result we need to introduce some notation. Let u_p be a radial solution with *m* nodal zones and

$$0 < r_{1,p} < r_{2,p} \cdots < r_{m,p} = 1$$
 be the zeros of u_p ,

 $A_{i,p}$ the nodal zones of u_p , precisely

 $A_{0,p} = \{x : |x| < r_{1,p}\}$, and $A_{i,p} = \{x : r_{i,p} < |x| < r_{i+1,p}\}$ for $i = 1, \dots, m-1$, $\mu_{i,p} = \max_{A_{i,p}} |u_p|$ the extremal value of $|u_p|$ in the $(i + 1)^{th}$ nodal zone $A_{i,p}$

 $\sigma_{i,p} \in A_{i,p}$ the extremal point of $|u_p|$ in the $(i+1)^{th}$ nodal zone,

so that
$$\mu_{i,p} = |\mu_{i,p}(\sigma_{i,p})|,$$

 $\widetilde{\mu}_{i,p} = (\mu_{i,p})^{\frac{p-1}{2+\alpha}},$
 $\widetilde{A}_{i,p} = \{x : x/\widetilde{\mu}_{i,p} \in A_{i,p}\}.$

For every i = 0, 1, ..., m - 1 we introduce the rescaled function

$$\widetilde{u}_{i,p}(x) := \frac{1}{\mu_{i,p}} \left| u_p\left(\frac{x}{\widetilde{\mu}_{i,p}}\right) \right| \quad \text{for } x \in \widetilde{A}_{i,p}, \tag{1.8}$$

231 Next, we let

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$$U_{\alpha}(x) := \left(1 + \frac{|x|^{2+\alpha}}{(N+\alpha)(N-2)}\right)^{-\frac{N-2}{2+\alpha}}$$
(1.9)

²³³ be the unique radial bounded solution of

l

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$$\begin{cases} -\Delta U_{\alpha} = |x|^{\alpha} U_{\alpha}^{p_{\alpha}} & \text{in } \mathbb{R}^{N}, \\ U_{\alpha} > 0 & \text{in } \mathbb{R}^{N}, \\ U_{\alpha}(0) = 1, \end{cases}$$
(1.10)

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see the "Appendix". Of course when $\alpha = 0$ (1.9) and (1.10) give back the well known Talenti

bubbles, which are related with problem (1.3).

...

237 Our main result on the asymptotic profile of radial solutions is the following:

Theorem 1.2 Let u_p be any radial solution to (1.1) with m nodal zones and $\alpha \ge 0$. When $p \rightarrow p_{\alpha}$ we have

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$$\mu_{t,p} \to \pm \infty, \qquad \qquad \text{for } t = 0, \dots m = 1, \qquad (1.11)$$

$$\widetilde{u}_{0,p} \to U_{\alpha} \text{ in } C^{1}_{\text{loc}}(\mathbb{R}^{N}) \qquad \qquad (1.12)$$

243 and whenever
$$m > 2$$

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$$r_{i,p} \to 0, \quad \sigma_{i,p} \to 0, \qquad for \, i = 1, \dots m - 1, \quad (1.13)$$

$$\widetilde{u}_{i,p} \to U_{\alpha} \quad in \ C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \qquad \qquad for \ i = 1, \dots m - 1. \tag{1.14}$$

The statements concerning i = 0 (i.e. the first nodal zone) follows easily by the already known results about the positive solution (see [2]), while the ones concerning the other nodal zones are far more delicate. The main source of difficulty is the supercritical setting,

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which can be overcome by performing a change of variable, introduced in [24], that allows 250 one to pass to a one-dimensional problem in a subcritical range. In this way the statement of 251 Theorem 1.2 becomes an extended radial version, in a noninteger dimension, of the analogous 252 one established in [18] for the Lane–Emden problem (i.e. when $\alpha = 0$). At that point the 253 most delicate part of the proof stands in establishing that the rescaled domains $A_{i,p}$ invade 254 \mathbb{R}^N , and this step requests a very fine knowledge of the speed of convergence (respectively, 255 divergence) of the zeros (respectively, extremal values) of the solution. The proof presented 256 here differs from the one in [18], even in the case $\alpha = 0$, because it does not rely on the 257 a-priori knowledge of the bubble tower shape of the radial solution. Indeed from our approach 258 it follows as a byproduct that for any $\alpha > 0$ radial nodal solutions of the Hénon problem 259 have a bubble tower shape with multiple blow up at the origin. 260

Another interesting consequence of the asymptotic analysis of the negative singular eigenvalues for the linearized operator L_u and of the characterization of the degeneracy of radial solutions given in [3] is the following result:

Theorem 1.3 Let u_p be any radial solution to (1.1) with m nodal zones and let $\alpha \ge 0$. Then there exists $\bar{p} \in (1, p_{\alpha})$ such that u is nondegenerate for any $p \in [\bar{p}, p_{\alpha})$.

Let us recall that a solution u is called nondegenerate whenever the linearized equation $L_u(v) = 0$ does not admit nontrivial solutions in $H_0^1(B)$. This consideration is new, even in the simpler case of the Lane–Emden problem, namely when $\alpha = 0$, and extends a previous result in this direction in [5] where it was proved that u is radially nondegenerate, namely that the linearized equation does not admit any radial solution.

To point out the usefulness of a nondegeneracy result as Theorem 1.3 we give here an easy application in proving existence results.

Theorem 1.4 Let $m \ge 1$ be any integer, either $\alpha = 0$ or $\alpha > 1$, and

$$\Omega_t := \{ x + t\sigma(x) : x \in B \},\$$

where $\sigma : \overline{B} \to \mathbb{R}^N$ is a smooth function, be a perturbation of the unit ball B. Then for every $p \in (\overline{p}, p_{\alpha})$ problem (1.1) settled in Ω_t admits a classical solution with m nodal zones, whose nodal set, when m > 1, does not touch the boundary $\partial \Omega_t$ for t small enough.

In the authors' opinion the existence result in Theorem 1.4 is interesting for two reasons. First because for $\alpha > 1$ it inherits a supercritical range, where the lack of variational setting makes more difficult to obtain existence of solutions. Secondly because it allows one to construct solutions shaped as the radial solutions without requiring any symmetry on Ω .

The paper is organized as follows. We start in Sect. 2 by proving the asymptotic profile of 282 nodal radial solutions to (1.1) with m nodal zones. In Sect. 3 we recall the characterization of 283 the Morse index of nodal radial solutions and we relate its computation to the computation 284 of the asymptotic limit of m negative singular radial eigenvalues as $p \to p_{\alpha}$. The analysis 285 of the first m-1 ones, outlined in Sect. 3.1, is based on the knowledge of the limit singular 286 eigenvalue problem and to an estimate previously obtained in [5]. The major difficulty is the 287 analysis of the last negative singular radial eigenvalue, performed in Sect. 3.2, which requires 288 some fine estimates that extend the previous one in the case of $\alpha = 0$. In Sect. 4 we prove 289 Theorems 1.3 and 1.4. Lastly we recall some well known fact about existence and uniqueness 290 of solutions for the limit problems in the "Appendix". 291

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

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2 The asymptotic profile of *u_p* via a "radially extended" version of the Lane–Emden problem

In this section we prove Theorem 1.2 by relating radial nodal solutions to (1.1) with nodal solutions to a radially extended version of the Lane Emden problem and studying the asymptotic behavior of these radially extended solutions. In order to distinguish the two radial solutions to (1.1) we will denote hereafter by u_p the nodal radial solution to (1.1) with *m* nodal zones, that satisfies

recalling that the other is given by the opposite of u_p .

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The proof of Theorem 1.2 will be given in a series of propositions in which we consider initially the first nodal zone, which is easier to handle, and then the case of the subsequent ones.

 u_p

To begin with, we furnish the proof of Theorem 1.2 for i = 0, which is an immediate consequence of the asymptotic behavior of positive radial solutions in [2] and does not rely on the radially extended Lane–Emden problem.

Proof of Theorem 1.2 for i = 0. Let us denote for a while by u_p^m the nodal radial solution to (1.1) with *m* nodal zones, that satisfies (2.1). It suffices to notice that, letting $r_{1,p}^m$ be the first zero of u_p^m , the scaled function

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 $\left(r_{1,p}^{m}\right)^{\frac{2+\alpha}{p-1}}u_{p}^{m}(r_{1,p}^{m}x)$

coincides with $u_p^1(x)$, the unique positive radial solution to the Hénon problem in B_1 . So applying [2, Proposition 3.6] gives (1.11) and (1.12) for i = 0.

The investigation of subsequent nodal zones is more delicate. An useful tool is the change of variables

$$\psi(t) = \left(\frac{2}{2+\alpha}\right)^{\frac{2}{p-1}} u(r), \quad t = r^{\frac{2+\alpha}{2}},$$
(2.2)

which has been introduced in [24] and transforms radial solutions to (1.1) into solutions of the radial extended Lane–Emden problem

$$\begin{cases} -\left(t^{M-1}v'\right)' = t^{M-1}|v|^{p-1}v, \quad 0 < t < 1, \\ v'(0) = 0, \quad v(1) = 0 \end{cases}$$
(2.3)

319 where

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$$M = M(N, \alpha) := \frac{2(N+\alpha)}{2+\alpha}$$
(2.4)

plays the role of a noninteger dimension. To deal with this problem we need to introduce the suitable functions spaces to which solutions to (2.3) belong. With this aim, for any $M, q \in \mathbb{R}, M \ge 2$ and $q \ge 1$, we let L_M^q be the weighted Lebesgue space of measurable functions $v : (0, 1) \to \mathbb{R}$ such that

$$\int_0^1 r^{M-1} |v|^q dr < +\infty$$

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(2.1)

Next we denote by H_M^1 the subspace of L_M^2 made up by that functions v which have weak 326 first order derivative in L_M^2 with 327

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and 329

 $H_{0M}^{1} = \{ v \in H_{M}^{1} : v(1) = 0 \}$

 $\int_0^1 r^{M-1} |v'|^2 dr < \infty,$

(2.5)

which is Hilbert space with the norm 331

$$\|v\|_M^1 := \left(\int_0^1 r^{M-1} (v')^2 dr\right)^{\frac{1}{2}}$$

due to a Poincaré inequality in the space $H_{0,M}^1$, see [3, Lemma 5.1]. The transformation (2.2) 333 maps $H^1_{0, rad}(B)$, the set of radial functions in $H^1_0(B)$, into $H^1_{0,M}$ with M as in (2.4) and can 334 be used in any dimension $N \ge 2$. It allows us to pass from u_p (the radial solution to (1.1) 335 with m nodal zones satisfying (2.1) to the solution to (2.3) with m nodal zones satisfying 336

$$v_p(0) > 0.$$
 (2.6)

The equivalence between radial solutions to (1.1) and solution to (2.3), both in classical 338 and weak sense and in any dimension N > 2, has been proved rigorously in [5, Corollary 339 4.2 and Lemma 4.3]. For the sake of completeness we recall that a weak radial solution to 340 (1.1) can be seen as $u \in H^1_{0,N}$ such that 341

$$\int_{0}^{1} r^{N-1} \left(u'\varphi' - r^{\alpha} |u|^{p-1} u\varphi \right) dr = 0$$
(2.7)

for any $\varphi \in H^1_{0,N}$, and similarly a weak solution to (2.3) is $v \in H^1_{0,M}$ such that 343

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 $\int_{0}^{1} t^{M-1} \left(v'\varphi' - |v|^{p-1}v\varphi \right) dt = 0$ (2.8)

for any $\varphi \in H^1_{0,M}$. In particular the same uniqueness result which holds for radial solutions 345 to (1.3) says that, for every integer $m \ge 1$, (2.3) admits a pair of solutions with m nodal 346 zones, which are one the opposite of the other and hence a unique solution v_p which satisfies 347 (2.6).348

Problem (2.3) can be seen as a "radially extended" version of the Lane-Emden problem 349 since when M is an integer v_p actually is the radial nodal solution to the Lane–Emden problem 350

$$\begin{cases} -\Delta v = |v|^{p-1}v & \text{in } B, \\ v = 0 & \text{on } \partial B, \end{cases}$$
(1.3)

settled in the unitary ball of \mathbb{R}^M . Also notice that when $N \ge 3$ then M > 2 and the threshold 353 exponent p_{α} of (1.1) can be expressed in term of the parameter $M = M(N, \alpha)$ as 354

$$p_{\alpha} = p_M = \frac{M+2}{M-2}.$$
 (2.9)

For integer $M \ge 3$, p_M is the critical value of the Sobolev immersion of $H_0^1(B)$ into $L^q(B)$, 357 and it constitutes the threshold for the existence of solutions for (1.3). For non integer M > 2358 the value p_M is still the critical exponent for the immersion of $H^1_{0,M}$ into L^q_M (see [3, Lemma 359 (2.3) and constitutes again the threshold for the existence of solutions to (2.3). 360

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

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We will give the proof of Theorem 1.2 in terms of the asymptotic behavior of the function $v_p \text{ as } p \rightarrow p_M$. For integer values of M this has been proved in [18, Propositions 3.3, 3.4 and Theorem 3.7]. Here we extend their result to any value of M > 2.

Let us first point out some qualitative property of the solutions v_p that shall be useful in the sequel, namely

Lemma 2.1 (Lemma 4.5 in [5]) Let $v_p \in H^1_{0,M}$ be the unique weak solution to (2.3) with m nodal zones that satisfies (2.6). Then $v_p \in C^2[0, 1]$ with

$$v_p(0) = \mathcal{M}_{0,p}, \quad v'_p(0) = 0.$$

Besides v_p is strictly decreasing in its first nodal zone and it has a unique critical point, $s_{i,p}$ in any nodal domain. In particular $s_{0,p} = 0$ is the global maximum point for v_p and for i = 1, ..., m - 1 it holds

$$\mathcal{M}_{0,p} = v_p(0) > \mathcal{M}_{1,p} = |v_p(s_{1,p})| > \dots \mathcal{M}_{m-1,p} = |v_p(s_{m-1,p})|.$$

In order to study its asymptotic profile as $p \rightarrow p_M$, we denote hereafter by

 $_{374} - 0 < t_{1,p} < t_{2,p} \cdots < t_{m,p} = 1$ the zeros of v_p ,

 $s_{75} - s_{0,p} = 0$ the extremal point of v_p in its first nodal zone $[0, t_{1,p})$,

- $s_{i,p}$ the extremal point of v_p in its $(i+1)^{th}$ nodal zone $(t_{i,p}, t_{i+1,p})$ for i = 1, ..., m-1, - $\mathcal{M}_{i,p} = (-1)^i v_p(s_{i,p})$ the extremal value of $|v_p|$ in the $(i+1)^{th}$ nodal zone, for i = 0

$$0, 1, \dots m$$

and, letting $t_{0,p} = 0$, we define the scaling

$$\widetilde{v}_{i,p}(t) = \frac{(-1)^i}{\mathcal{M}_{i,p}} v_p\left(\frac{t}{\widetilde{\mathcal{M}}_{i,p}}\right) \quad \text{as } t_{i,p} < \frac{t}{\widetilde{\mathcal{M}}_{i,p}} < t_{i+1,p}, \tag{2.10}$$

382 for i = 0, ..., m - 1. Here

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 $\widetilde{\mathcal{M}}_{i,p} = \left(\mathcal{M}_{i,p}\right)^{\frac{p-1}{2}}.$ (2.11)

These newly introduced items are related to the respective ones for the Hénon problem by the following relations

 $-r_{i,p} = (t_{i,p})^{\frac{2}{2+\alpha}} \text{ are the zeros of } u_p,$ $-\mu_{i,p} = (\frac{2+\alpha}{2})^{\frac{2}{p-1}} \mathcal{M}_{i,p} \text{ are the local extremal values of } u_p,$ $-\sigma_{i,p} = (s_{i,p})^{\frac{2}{2+\alpha}} \text{ are extremal values of } u_p,$

$$\widetilde{u}_{i,p}(r) = \widetilde{v}_{i,p}\left(\frac{2}{2+\alpha}r^{\frac{2+\alpha}{2}}\right).$$
(2.12)

It is easy to check that, for i = 0, ..., m - 1 the functions $\tilde{v}_{i,p}$ solves

$$\begin{cases} -(t^{M-1}\widetilde{v}'_{i,p})' = t^{M-1}\widetilde{v}^p_{i,p}, & \text{for } t_{i,p}\widetilde{\mathcal{M}}_{i,p} < t < t_{i+1,p}\widetilde{\mathcal{M}}_{i,p} \\ \widetilde{v}_{i,p}\left(t_{i+1,p}\widetilde{\mathcal{M}}_{i,p}\right) = 0 & \\ \widetilde{v}_{i,p}\left(t_{i,p}\widetilde{\mathcal{M}}_{i,p}\right) = 0 & \text{when } i \ge 1 \end{cases}$$

$$(2.13)$$

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For simplicity we will assume that $\tilde{v}_{i,p}$ is defined on $(0, \infty)$ extending it to zero outside the interval $(t_{i,p}\tilde{\mathcal{M}}_{i,p}, t_{i+1,p}\tilde{\mathcal{M}}_{i,p})$.

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 $\begin{cases} -\left(t^{M-1}V'\right)' = t^{M-1}V^{p_M}, & t > 0, \\ V(t) > 0 & t > 0, \end{cases}$

with the condition 397

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V(0) = 1

whose unique weak solution in the space $\mathcal{D}_M(0,\infty)$ is given by 399

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$$V_M(t) = \left(1 + \frac{t^2}{M(M-2)}\right)^{-\frac{M-2}{2}}$$
(2.16)

see the "Appendix". Here $\mathcal{D}_M(0,\infty)$ stands for the closure of $C_0^{\infty}[0,\infty)$ under the norm 401

$$\int_0^\infty r^{M-1} |v'|^2 dr,$$

which is a natural generalization of the space $D_{rad}^{1,2}(\mathbb{R}^N)$ to the case of the non-integer dimen-403 sion M, and by weak solution to (2.14) we mean a function $V \in \mathcal{D}_M(0, \infty)$ such that 404

$$\int_0^\infty r^{M-1} V' \varphi' \, dr = \int_0^\infty r^{M-1} V^{p_M} \varphi \, dr$$

for every $\varphi \in \mathcal{D}_M(0, \infty)$.

Since we have already proved that for i = 0 the statements of Theorem 1.2 hold true, it 407 remains to consider the case of $i \ge 1$. Concerning the subsequent nodal zones Theorem 1.2 408 is equivalent to the two following propositions 409

Proposition 2.2 For any M > 2, for any integer $m \ge 2$ and i = 1, ..., m - 1 we have 410

$$\mathcal{M}_{i,p} \to +\infty, \tag{2.17}$$

$$i, p \to 0, \quad t_{i, p} \to 0, \tag{2.18}$$

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as $p \rightarrow p_M$ given by (2.9). 414

- **Proposition 2.3** For any M > 2, for any integer $m \ge 2$ and i = 1, ..., m 1 we have 415
 - $\widetilde{v}_{i,p} \to V_M \quad in \ C^1_{\text{loc}}(0,+\infty)$ (2.19)

as $p \rightarrow p_M$ given by (2.9). 418

Indeed assuming Propositions 2.2 and 2.3 one can easily deduce that the statement of 419 Theorem 1.2 holds true for $i = 1, \ldots m - 1$. 420

Proof of Theorem 1.2 for i = 1, ..., m - 1. Equations (2.17) and (2.18) immediately give 421

(1.11) and (1.13), recalling the relations between $t_{i,p}$, $s_{i,p}$ and $\mathcal{M}_{i,p}$ and $r_{i,p}$, $\sigma_{i,p}$ and $\mu_{i,p}$. 422

Similarly (1.14) follows by (2.19) thanks to (2.12). 423

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

(2.14)

(2.15)

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2.1 The proof of Propositions 2.2 and 2.3 424

In this subsection we will prove the two propositions which give the asymptotic behavior of the 425 function v_p as $p \to p_M$. Proposition 2.3 will be proved passing to the limit into (2.13), which 426 is possible because (i) the functions $\tilde{v}_{i,p}$, extended to zero outside $(t_{i,p}\mathcal{M}_{i,p}, t_{i+1,p}\mathcal{M}_{i,p})$, 427 are uniformly bounded in $\mathcal{D}_M(0,\infty)$, and (ii) $t_{i,p}\mathcal{M}_{i,p} \to 0$ while $t_{i+1,p}\mathcal{M}_{i,p} \to \infty$. Item 428 (ii) is the most delicate part of the proof and requests a deep knowledge of the behavior of 429 the zeros and of the extremal values of the function v_p . Proposition 2.2 is a first step in this 430 direction and it has been put in evidence because it has interest in itself. In any case the proof 431 of these facts is quite involved and requires some preliminary estimates. 432

This first lemma provides a bound on the energy of the solution v_p in each nodal zone 433 and a bound on the first derivate of v_p which will be useful in the sequel in order to pass to 434 the limit into (2.13). 435

Lemma 2.4 There exist $\delta > 0$ and constant C_1, C_2 such that for every $p \in (1 + \delta, p_M)$ 436

$$\int_{t_{i-1,p}}^{t_{i,p}} t^{M-1} |v_p'|^2 dt = \int_{t_{i-1,p}}^{t_{i,p}} t^{M-1} |v_p|^{p+1} dt \le C_1,$$
(2.20)

for any $i = 1, \ldots m$ and 439

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$$|v_p'(t)| \le C_2 t^{\frac{2-p(M-2)}{2}}$$
(2.21)

442 as
$$t \in (0, 1)$$
.

Proof Using as a test function in (2.8) the function which coincides with v_p on $(t_{i-1,p}, t_{i,p})$ 443 and is zero elsewhere immediately gives the first equality in (2.20). The subsequent estimate 444 follows by the Nehari construction. Indeed the solution v_p can be produced by solving the 445 minimization problem 446

447
$$\Lambda(t_1, \cdots t_{m-1}) = \min \left\{ \sum_{i=0}^{m-1} \inf_{\phi_i \in \mathcal{N}(t_i, t_{i+1})} \mathcal{E}(\phi_i) : 0 = t_0 < t_1 < \cdots < t_m = 1 \right\},$$

where $\mathcal{N}(t_i, t_{i+1})$ are the Nehari manifolds 449

$$\mathcal{N}(t_i, t_{i+1}) = \left\{ \phi \in H^1_{0,M} : \phi(r) = 0 \text{ for } r \text{ outside } (t_i, t_{i+1}), \right.$$

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 $|\phi'|^2 dr = \int_{t_i} r^{M-1} |\phi|^{p+1} dr \Big\},\,$ 452

and \mathcal{E} stands for the energy functional 453

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$$\mathcal{E}(\phi) = \frac{1}{2} \int_0^1 r^{M-1} |v'|^2 dr - \frac{1}{p+1} \int_0^1 r^{M-1} |v|^{p+1} dr.$$

Then it can be checked that choosing t_1, \ldots, t_{m-1} which realize the minimum Λ and gluing 456 together, alternatively, the positive and negative solution in the sub-interval (t_{i-1}, t_i) , gives 457 a nodal solution to (2.3), which by uniqueness, see [31], coincides with v_p up to the sign. 458 We refer both to [10] and [5, Sec. 4] for more details. For the current purpose it suffices to 459 notice that for all $i = 0, \dots, m-1$ the restrictions $v_{i,p}$ of the solution v_p to its nodal zones 460

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 (t_i, t_{i+1}) belong to the Nehari sets $\mathcal{N}(t_i, t_{i+1})$ and therefore 461

$$\int_{0}^{1} r^{M-1} |v'_{p}|^{2} dr = \sum_{i=0}^{m-1} \int_{t_{i}}^{t_{i+1}} r^{M-1} |v'_{i,p}|^{2} dr = \frac{2(p+1)}{p-1} \Lambda(t_{1}, \dots t_{m-1})$$
$$\leq \frac{2(p+1)}{p-1} \sum_{i=0}^{m-1} \mathcal{E}(\phi_{i,p}) = \sum_{i=0}^{m-1} \int_{\frac{i}{m}}^{\frac{i+1}{m}} r^{M-1} |\phi'_{i,p}|^{2} dr$$

464 for every *m*-ple of functions $\phi_{i,p} \in \mathcal{N}(\frac{i}{m}, \frac{i+1}{m})$ and for every $p \in (1, p_M)$. So (2.20) can be proved by producing a sequence $\phi_{i,p}$ with 465

$$\lim_{p \to p_M} \int_{\frac{i}{m}}^{\frac{i+1}{m}} r^{M-1} |\phi_{i,p}'|^2 dr < +\infty \quad \text{for } i = 0, \dots m-1.$$
(2.22)

To this aim we take continuous piecewise linear functions defined as 468

$$\phi_{i,p}(r) = \begin{cases} a_{i,p} \left(r - \frac{i}{m} \right) & \text{as } \frac{i}{m} < r \le \frac{2i+1}{2m}, \\ a_{i,p} \left(\frac{i+1}{m} - r \right) & \text{for } \frac{2i+1}{2m} < r < \frac{i+1}{m}, \\ 0 & \text{elsewhere} \end{cases}$$

and pick $a_{i,p} > 0$ in such a way that $\phi_{i,p} \in \mathcal{N}(\frac{i}{m}, \frac{i+1}{m})$. Since 470

$$\int_{\frac{i}{m}}^{\frac{i+1}{m}} r^{M-1} |\phi_{i,p}'|^2 dr = a_{i,p}^2 \int_{\frac{i}{m}}^{\frac{i+1}{m}} r^{M-1} dr = \frac{a_{i,p}^2}{m^M} \int_{0}^{1} (i+r)^{M-1} dr,$$

$$\int_{\frac{i}{m}}^{\frac{i+1}{m}} r^{M-1} |\phi_{i,p}|^{p+1} dr = a_{i,p}^{p+1}$$

$$\int \frac{1}{m} dr = \frac{a_{i,p}^{p+1}}{m^{M+p+1}} \int_{0}^{\frac{1}{2}} \left((i+r)^{M-1} + (i+1-r)^{M-1} \right) r^{p+1} dr$$

 $\phi_{i,p} \in \mathcal{N}(\frac{i}{m}, \frac{i+1}{m})$ provided that 476

$$a_{i,p}^{p-1} = \frac{m^{p+1} \int_0^1 (i+r)^{M-1} dr}{\int_0^{\frac{1}{2}} \left((i+r)^{M-1} + (i+1-r)^{M-1} \right) r^{p+1} dr},$$
 and in that case

478 and in that case

$$\int_{\frac{i}{m}}^{\frac{i+1}{m}} r^{M-1} |\phi_{i,p}'|^2 dr = \frac{m^{2\frac{p+1}{p-1}-M} \left(\int_0^1 (i+r)^{M-1} dr\right)^{\frac{p+1}{p-1}}}{\left(\int_0^{\frac{1}{2}} \left((i+r)^{M-1} + (i+1-r)^{M-1}\right) r^{p+1} dr\right)^{\frac{2}{p-1}}},$$

which clearly yields (2,22). 480

Besides from (2.20) and the Talenti's Sobolev embedding for the spaces $H_{0,M}^1$ stated by 481 [3, Lemma 5.3] we also get 482

$$\left(\int_{0}^{1} t^{M-1} |v_{p}|^{\frac{2M}{M-2}} dt\right)^{\frac{2}{2^{*}_{M}}} \leq S_{M} \int_{0}^{1} t^{M-1} |v_{p}'|^{2} dt \leq C.$$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

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Author Proof

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Next integrating Eq. (2.3) on (0, t) and using that $v'_p(0) = 0$ give

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488 Eventually Holder inequality yields

$$\begin{aligned} |v_{p}'(t)| &\leq \frac{1}{t^{M-1}} \left(\int_{0}^{t} \tau^{M-1} |v_{p}|^{\frac{2M}{M-2}} d\tau \right)^{\frac{p(M-2)}{2M}} \left(\int_{0}^{t} \tau^{M-1} d\tau \right)^{1 - \frac{p(M-2)}{2M}} \\ &\leq \frac{1}{t^{M-1}} C t^{M - \frac{p(M-2)}{2}} = C t^{1 - \frac{p(M-2)}{2}}. \end{aligned}$$

 $|v_p'(t)| \le \frac{1}{t^{M-1}} \int_0^t \tau^{M-1} |v_p|^p d\tau.$

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⁴⁹³ Next lemma shows that the energy of v_p is bounded also from below in each nodal zone, ⁴⁹⁴ and so ensures that the local extremal values do not vanish.

495 **Lemma 2.5** For all i = 0, ..., m - 1 we have

$$\liminf_{p \to p_M} \int_{t_{i,p}}^{t_{i+1,p}} t^{M-1} |v_p|^{p+1} dt = \liminf_{p \to p_M} \int_{t_{i,p}}^{t_{i+1,p}} t^{M-1} |v_p'|^2 dt \ge S_M^{\frac{M}{2}}$$

497 In particular $\liminf_{p \to p_M} \mathcal{M}_{i,p} > 0.$

Here S_M is the best constant for the Sobolev embedding of $H_{0,M}^1$ into $L_M^{2^*_M}$, with $2^*_M = \frac{2M}{M-2}$ (see [3, Lemma 5.4]).

⁵⁰⁰ **Proof** Since $\lim_{p \to p_M} \frac{p+1}{p-1} = \frac{M}{2}$, it suffices to show that

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 $\liminf_{p \to p_M} \left(\int_{t_{i,p}}^{t_{i+1,p}} t^{M-1} (v'_p)^2 dt \right)^{\frac{p-1}{p+1}} \ge S_M.$

We use as a test function in (2.8) the function $v_{i,p}$ which coincides with v_p in the set $(t_{i,p}, t_{i+1,p})$ and it is zero elsewhere, obtaining that

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$$\int_{t_{i,p}}^{t_{i+1,p}} t^{M-1} (v'_p)^2 dt = \int_0^1 t^{M-1} (v'_{i,p})^2 dt = \int_0^1 t^{M-1} |v_{i,p}|^{p+1} dt = \int_{t_{i,p}}^{t_{i+1,p}} t^{M-1} |v_p|^{p+1} dt.$$

505 Hence

$$\left(\int_{t_{i,p}}^{t_{i+1,p}} t^{M-1}(v'_{i,p})^2 dt\right)^{\frac{p-1}{p+1}} = \frac{\int_0^1 t^{M-1}(v'_{i,p})^2 dt}{\left(\int_0^1 t^{M-1}|v_{i,p}|^{p+1} dt\right)^{\frac{2}{p+1}}} \stackrel{\geq}{\underset{\text{Holder}}{\overset{\int_0^1 t^{M-1}(v'_{i,p})^2 dt}{\frac{\int_0^1 t^{M-1}(v'_{i,p})^2 dt}{M^{\frac{2}{2_M^2} - \frac{2}{p+1}} \left(\int_0^1 t^{M-1}|v_{i,p}|^{2_M^*} dt\right)^{\frac{2}{2_M^*}}}} \ge M^{\frac{2}{p+1} - \frac{2}{2_M^*}} S_M,$$

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where the last inequality holds thanks to the Talenti's Sobolev embedding, [36], see also [3, Lemma 5.4]. The first part of the claim follows because $\frac{2}{p+1} - \frac{2}{2_{M}^{*}} \rightarrow 0$.

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To conclude the proof it suffices to notice that, due to Lemma 2.1,

$$(m-i)S_{M}^{\frac{M}{2}} \leq \liminf_{p \to p_{M}} \int_{t_{i,p}}^{1} t^{M-1} |v_{p}(t)|^{p+1} dt \leq \liminf_{p \to p_{M}} (\mathcal{M}_{i,p})^{p+1} (1-t_{i,p}).$$
(2.23)

As a corollary of the previous lemmas we obtain the boundedness of $\tilde{v}_{i,p}$ in $\mathcal{D}_M(0, +\infty)$.

Corollary 2.6 For i = 0, ..., m - 1 let $\tilde{v}_{i,p}$ be the rescaled function defined in (2.10) and extended to zero outside $(t_{i,p}, t_{i+1,p})$. Then there exists $\delta > 0$ and a constant C_3 such that

$$\int_0^\infty t^{M-1} \left(\widetilde{v}'_{i,p}\right)^2 dt \le C_3 \tag{2.24}$$

518 for every $p \in (p_M - \delta, p_M)$.

519 **Proof** It is enough to observe that

$$\int_{0}^{\infty} t^{M-1} \left(\widetilde{v}'_{i,p}\right)^{2} dt = \int_{t_{i-1,p},\widetilde{\mathcal{M}}_{i,p}}^{t_{i,p},\widetilde{\mathcal{M}}_{i,p}} t^{M-1} \left(\widetilde{v}'_{i,p}\right)^{2} dt$$
$$= \mathcal{M}_{i,p}^{\frac{p-1}{2}(M-2)-2} \int_{t_{i-1,p}}^{t_{i,p}} t^{M-1} |v'_{p}|^{2} dt \leq C_{3}$$

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by (2.20), since
$$\frac{p-1}{2}(M-2) < 2$$
 and $\mathcal{M}_{i,p} \ge \varepsilon > 0$ by Lemma 2.5.

We also recall a fine estimate of the behavior of the function v_p in a left neighborhood of its zeros, which is fundamental in the computations.

Lemma 2.7

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$$|v_p(t)| \leq \frac{\mathcal{M}_{0,p}}{\left(1 + \frac{(\widetilde{\mathcal{M}}_{0,p}t)^2}{M(M-2)}\right)^{\frac{M-2}{2}}}$$

527 for every $0 \le t < t_{1,p}$.

⁵²⁸ Moreover if $s_{i,p}/t_{i+1,p} \to 0$ for some i = 1, ..., m-1, then for any $\varepsilon \in (0, 1)$ there exist ⁵²⁹ $\gamma = \gamma(\varepsilon) > 1$ and $\bar{p} = \bar{p}(\varepsilon) < p_M$ such that

$$|v_p(t)| \le \frac{\mathcal{M}_{i,p}}{\left(1 + \frac{\varepsilon(\widetilde{\mathcal{M}}_{i,p}t)^2}{M(M-2)}\right)^{\frac{M-2}{2}}}$$
(2.25)

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531 for every $\gamma s_{i,p} \leq t \leq t_{i+1,p}$ and $p \in (\bar{p}, p_M)$.

The first part of the statement, concerning the first nodal zone, can be proved by performing the Emden-Fowler transformation and following the line of [7], see also [20, Lemma 2], where the same estimate is obtained for positive solutions. Next their arguments can be adapted to deal with the subsequent nodal zones, as it has been done in [18, Propositions 3.5 and 3.6], where the same statement of Lemma 2.7 was proved only for integer *M*. Their proof applies to any M > 2 because it only makes use of ODE arguments.

Let us remark that the previous estimates can be read in terms of the scaled functions $\tilde{v}_{i,p}$ as follows

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Corollary 2.8

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Author Proof

$$\widetilde{v}_{0,p}(t) \leq V_M(t) \quad \text{for every } 0 \leq t < t_{1,p} \widetilde{\mathcal{M}}_{0,p}.$$

Moreover if $s_{i,p}/t_{i+1,p} \to 0$ for some i = 1, ..., m - 1, then for any $\varepsilon \in (0, 1)$ there exist $\gamma = \gamma(\varepsilon) > 1$ and $\bar{p} = \bar{p}(\varepsilon) < p_M$ such that

$$\widetilde{v}_{i,p}(t) \le V_M(\sqrt{\varepsilon}t) \quad \text{for every } \gamma s_{i,p} \widetilde{M}_{i,p} < t < t_{i+1,p} \widetilde{M}_{i,p}$$
(2.26)

544 as $p \in (\bar{p}, p_M)$.

Proposition 2.2 will be proved proceeding forward from the first nodal zone to the second one and so on. Hence the starting point stands in describing the asymptotics of $\tilde{v}_{0,p}$ in the first nodal zone, which is a consequence of Theorem 1.2 for i = 0 and has been already proved. Precisely the part of the statement concerning the first nodal zone is equivalent to

Proposition 2.9 For every M > 2 and any integer $m \ge 1$, $\mathcal{M}_{0,p} \to +\infty$ and $\tilde{v}_{0,p} \to V_M$ in $C^1_{\text{loc}}[0, +\infty)$, as $p \to p_M$.

⁵⁵¹ **Proof** It suffices to take $\alpha > 0$ such that $N = M + \alpha(M/2 - 1)$ is an integer and then apply ⁵⁵² Theorem 1.2, which has already been proved at the beginning of this section in the particular ⁵⁵³ case i = 0.

It will also be needed to establish relations between the asymptotics of the extremal values in different nodal zones. To this aim we introduce another scaling of the solution v_p that we will use later on, precisely

$$w_{i,p}(r) = (t_{i,p})^{\frac{2}{p-1}} v_p(t_{i,p} r), \qquad (2.27)$$

558 which satisfies

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557

$$\begin{cases} -\left(r^{M-1}w'_{i,p}\right)' = r^{M-1}|w_{i,p}|^{p-1}w_{i,p} & \text{for } 0 < r < 1/t_{i,p}, \\ w'_{i,p}(0) = w_{i,p}(1) = 0 = w_{i,p}(1/t_{i,p}). \end{cases}$$
(2.28)

We therefore see that $w_{i,p}$ on the interval (0, 1) coincides with the nodal solution to (2.3) which has exactly *i* nodal zones, but is defined also in the larger interval $(0, 1/t_{i,p})$. This will be of help when deducing the asymptotics of the extremal value in the *i*th nodal zone from the one in the previous nodal zone. We deal by now with the behavior of the function $w_{i,p}$ to the left of r = 1.

Lemma 2.10 Take i = 1, ..., m - 1 and assume that, for a sequence $p_n \rightarrow p_M$,

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$$\tau_n := s_{i-1,p_n}/t_{i,p_n} \to 0$$

$$\rho_n := t_{i,p_n} \widetilde{\mathcal{M}}_{i-1,p_n} \to +\infty.$$
(2.29)
(2.29)
(2.30)

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Then $w_{i,p_n} \to 0$ uniformly in any set $[1 - \delta, 1]$ for $0 < \delta < 1$.

Proof For simplicity of notation we shall write w_n and t_n instead of w_{i,p_n} and t_{i,p_n} . By Lemma 2.7, for a fixed $\varepsilon > 0$ there exists γ such that

$$|w_{n}(r)| \leq \frac{\rho_{n}^{\frac{p}{p_{n}-1}}}{\left(1 + \frac{\varepsilon(\rho_{n}r)^{2}}{M(M-2)}\right)^{\frac{M-2}{2}}} \quad \text{for } \gamma \tau_{n} \leq r \leq 1.$$
(2.31)

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

If $\delta \in (0, 1)$ is fixed, by hypothesis (2.29) there exists \bar{n} such that $\gamma \tau_n \le 1 - \delta$ if $n \ge \bar{n}$ and (2.31) implies that for any $r \in [1 - \delta, 1]$ we have

$$|w_n(r)| \le C(\varepsilon, \delta) \rho_n^{\frac{2}{p_n - 1} - M + 2} = o(1)$$

as $n \to \infty$ by (2.30), because $\frac{2}{p_n - 1} - M + 2 \to -\frac{M - 2}{2} < 0$.

We are now in the position to prove Proposition 2.2.

⁵⁷⁸ **Proof of Proposition 2.2** It is worth noticing by now that the Radial Lemma in $H_{0,M}^1$ (see [3, Lemma 5.2]) yields

$$t_{i,p} < s_{i,p} \le \left(\frac{\int_0^1 t^{M-1} \left| v_p'(t) \right|^2 dt}{(M-2) \left(\mathcal{M}_{i,p} \right)^2} \right)^{\frac{1}{M-2}} \le \frac{C}{\left(\mathcal{M}_{i,p} \right)^{\frac{2}{M-2}}},$$

So, once (2.17) has been established, then both $t_{i,p}$ and $s_{i,p}$ go to zero, which means that 581 the proof is completed. Besides it is already known by Proposition 2.9 that $\mathcal{M}_{0,p} \to +\infty$, 582 therefore (2.17) can be proved by assuming $\mathcal{M}_{i-1,p} \to +\infty$ and deducing that also $\mathcal{M}_{i,p} \to$ 583 $+\infty$. To this aim we assume by contradiction that $\mathcal{M}_{i,p}$ is bounded, so that the functions 584 v_p are uniformly bounded in $[t_{i,p}, 1]$ by Lemma 2.1. Up to a subsequence we may assume 585 $\mathcal{M}_{i,p} \to \bar{\mathcal{M}} \in (0, +\infty)$ and $t_{i,p} \to T \in [0, 1)$. Indeed the occurrence $\bar{\mathcal{M}} = 0$ is ruled out 586 by Lemma 2.5, and T = 1 is not allowed by (2.23) since we are assuming $\mathcal{M}_{i,p}$ bounded. 587 Next we argue separately according to wheter 588

589 a)
$$T = 0$$
,

590 b) or $T \in (0, 1)$.

In case a) we observe that the functions v_p are bounded in $H_{0,M}^1$ by (2.20). So, up to a subsequence, v_p converges to a function \bar{v} weakly in $H_{0,M}^1$, and also strongly in L_M^q for every $1 < q < \frac{2M}{M-2}$ by the compact Sobolev embedding stated in [3, Lemma 5.4]. It is thus easy to see that we can pass to the limit in (2.8) so that $\bar{v} \in H_{0,M}^1$ is a weak solution to

$$\begin{cases} -\left(t^{M-1}\bar{v}'\right)' = t^{M-1}|\bar{v}|^{p_M-1}\bar{v} \quad t \in (0,1), \\ \bar{v}(1) = 0. \end{cases}$$

⁵⁹⁶ Next we denote by $\hat{v}_{i,p}$ the function which coincides with v_p on $(t_{i,p}, 1)$ and is null on ⁵⁹⁷ $[0, t_{i,p}]$. Since we are assuming that $\mathcal{M}_{i,p}$ remains bounded, Lemma 2.1 assures that $\hat{v}_{i,p}$ ⁵⁹⁸ is uniformly bounded on [0, 1] and clearly it converges pointwise a.e. to \bar{v} because we are ⁵⁹⁹ assuming $t_{i,p} \to 0$. So we can pass to the limit and compute

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$$\int_{0}^{1} t^{M-1} |\bar{v}|^{p_{M}+1} dt = \lim_{p \to p_{M}} \int_{0}^{1} t^{M-1} |\hat{v}_{i,p}|^{p+1} dt$$
$$= \lim_{p \to p_{M}} \int_{t_{i,p}}^{1} t^{M-1} |v_{p}|^{p+1} dt \ge (m-i) S_{M}^{\frac{M}{2}}$$

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⁶⁰³ by Lemma 2.5. Hence \bar{v} is not trivial. Eventually performing the change of variables (2.2) ⁶⁰⁴ backwards (and invoking [3, Proposition 4.5]) gives a nontrivial radial solution of the Hénon ⁶⁰⁵ problem in a ball with the exponent p_{α} , which is not possible by Pohozaev identity.

In case b), we look at the function $w_{i,p}$ introduced in (2.27). In the present setting $\tau_p = s_{i-1,p}/t_{i,p} \to 0$ and $\rho_p = t_{i,p}\widetilde{\mathcal{M}}_{i-1,p} \to \infty$ (since we are assuming $s_{i-1,p} \to 0$, $t_{i,p} \to 0$).

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

⁶⁰⁸ $T \neq 0$, $\widetilde{\mathcal{M}}_{i-1,p} \to \infty$), so Lemma 2.10 implies that $w_{i,p} \to 0$ uniformly on any set of type ⁶⁰⁹ $[1 - \delta, 1]$. In particular $w_{i,p}$ is uniformly bounded on $[1 - \delta, 1]$. But the same holds also in ⁶¹⁰ the set $[1, 1/t_{i,p}]$, because in that case

$$|w_{i,p}(r)| = t_{i,p}^{\frac{2}{p-1}} |v_p(t_{i,p}r)| \le \mathcal{M}_{i,p}$$

612 Moreover

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$$|w'_{i,p}(r)| \le |v'_p(t_{i,p}r)| \le C \quad \text{in} \ (1-\delta, 1/t_{i,p}) \tag{2.32}$$

thanks to (2.21), since we are assuming that $t_{i,p}$ does not vanish. Next using the fact that w_p is a classical solution to (2.28) one sees that also $|w_{i,p}''| \le C$ in $(1 - \delta, 1/t_{i,p})$ so that, up to a subsequence, $w_{i,p}$ converges in $C^1(1 - \delta, 1/T)$ to a function w which weakly solves

$$\begin{cases} -(t^{M-1}w')' = t^{M-1}|w|^{p_M-1}w & \text{for } 1-\varepsilon < t < 1/T, \\ w(1) = 0 = w(1/T). \end{cases}$$

⁶¹⁸ Next by [3, Corollary 4.8] a weak solution w is also classical. As we already noticed that ⁶¹⁹ w = 0 on the interval $(1 - \delta, 1)$, the unique continuation principle gives that w is identically ⁶²⁰ zero. But this contradicts Lemma 2.5 since by the boundedness of $w_{i,p}$

$$0 = \int_{1}^{1/T} t^{M-1} |w|^{p_M+1} dt = \liminf_{p \to p_M} \int_{1}^{1/t_{i,p}} t^{M-1} |w_{i,p}|^{p+1} dt$$

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$$= \liminf_{p \to p_M} (t_{i,p})^{\frac{2(p+1)}{p-1} - M} \int_{t_{i,p}}^1 t^{M-1} |v_p|^{p+1} dt \ge (m-i) S_M^{\frac{M}{2}}$$

because $\frac{2(p+1)}{p-1} - M \to 0$ and $t_{i,p} \to T \in (0, 1)$. So neither item b) can happen and the proof is completed.

Eventually we prove Proposition 2.3.

627 Proof of Proposition 2.3 Assume for a while to know that

$$t_{i+1,p}\,\tilde{\mathcal{M}}_{i,p}\to+\infty,\tag{2.33}$$

$$s_{i,p} \widetilde{\mathcal{M}}_{i,p} \to 0, \tag{2.34}$$

$$t_{i,p} \widetilde{\mathcal{M}}_{i,p} \to 0, \qquad (2.35)$$

as $p \to p_M$, for i = 1, ..., m - 1. Then it is not hard conclude the proof. As the nodal domain $(t_{i,p} \,\widetilde{\mathcal{M}}_{i,p}, t_{i+1,p} \,\widetilde{\mathcal{M}}_{i,p})$ invades $(0, +\infty)$, it is equivalent to prove the convergence of the sequence of functions $\widetilde{v}_{i,p}$ extended to be zero outside $(t_{i,p} \,\widetilde{\mathcal{M}}_{i,p}, t_{i+1,p} \,\widetilde{\mathcal{M}}_{i,p})$ so that they belong to $\mathcal{D}_M(0, \infty)$. We recall that $\widetilde{v}_{i,p}$ is nonnegative and solves the Eq. (2.13) in classical sense. Moreover its norm in $\mathcal{D}_M(0, \infty)$ is bounded (uniformly w.r.t. p) by Corollary 2.6 therefore $\widetilde{v}_{i,p}$ converges to a function \widetilde{v} weakly in $\mathcal{D}_M(0, \infty)$, strongly in $L^q(0, \infty)$ as $q = 2_M^*$ and pointwise a.e., up to a subsequence.

We can then pass to the limit in the weak formulation of (2.13), provided that the functions $r^{M-1}\widetilde{v}_{i,p}^{p}$ are uniformly dominated by a function in $L^{1}(0, \infty)$ (for *p* close to p_{M}). First observe that we can apply Corollary 2.8 thanks to assumptions (2.33) and (2.34). More precisely we know that for a fixed $\varepsilon > 0$ there exist $\gamma > 0$ and $\bar{p} \in (1, p_{M})$ such that for every $p \in (\bar{p}, p_{M})$ and $r \in (\gamma s_{i,p} \widetilde{\mathcal{M}}_{i,p}, t_{i+1,p} \widetilde{\mathcal{M}}_{i,p})$

$$\widetilde{v}_{i,p}(r) \leq V_M(\sqrt{\varepsilon}r)$$

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and, recalling that $\widetilde{v}_{i,p} = 0$ when $r > t_{i+1,p} \widetilde{\mathcal{M}}_{i,p}$ and $\gamma s_{i,p} \widetilde{\mathcal{M}}_{i,p} \to 0$, taking eventually a 645 larger value of p, we have for every r > 1646

$$r^{M-1} \left| \widetilde{v}_{i,p}(r) \right|^{p} \le r^{M-1} \left(V_{M}(\sqrt{\varepsilon}r) \right)^{p} = r^{M-1} \left(1 + \frac{\varepsilon r^{2}}{M(M-2)} \right)^{-\frac{M-2}{2}p} < Cr^{M-1-(M-2)p}$$

which belongs to $L^1(1,\infty)$ for $p > \frac{M}{M-2}$. For $r \in (0,1)$, instead we have 650

$$r^{M-1}\left(\widetilde{v}_{i,p}(r)\right)^p \le 1$$

by construction. Then it is easy to see that the limit function \tilde{v} is a weak solution to the 652 equation in (2.14). 653

Eventually one can see that the limit function \tilde{v} is not null and satisfies $\tilde{v}(0) = 1$, and so 654 that it coincides with the function V_M identified by (2.16), see also the "Appendix". This can 655 be seen by using the same arguments of [27, Lemma 6]. Indeed $s_{i,p}\mathcal{M}_{i,p}$ is a critical point 656 for $\tilde{v}_{i,p}$ and integrating (2.13) on the interval between $s_{i,p} \mathcal{M}_{i,p}$ and t gives 657

$$\widetilde{v}_{i,p}'(t) = -t^{1-M} \int_{s_{i,p}\widetilde{\mathcal{M}}_{i,p}}^{t} r^{M-1} \widetilde{v}_{i,p}^{p} dr \text{ for } t_{i,p} \widetilde{\mathcal{M}}_{i,p} < t < t_{i+1,p} \widetilde{\mathcal{M}}_{i,p}.$$
(2.36)

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Moreover for every r > 0 (2.34) assures that $s_{i,p}\widetilde{\mathcal{M}}_{i,p} < r$ for p near p_M and so (2.36) 659 gives 660

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$$\widetilde{v}'_{i,p}(r) = -r^{1-M} \int_{s_{i,p}\widetilde{\mathcal{M}}_{i,p}} t^{M-1} \widetilde{v}^p_{i,p} dt \ge -r^{1-M} \int_{s_{i,p}\widetilde{\mathcal{M}}_{i,p}} t^{M-1} dt$$
$$= -\frac{r}{M} \left(1 - \left(\frac{s_{i,p}\widetilde{\mathcal{M}}_{i,p}}{r}\right)^M \right) \ge -\frac{r}{M}.$$

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Then, recalling that $\widetilde{v}_{i,p}(s_{i,p}\widetilde{\mathcal{M}}_{i,p}) = 1$, we have 664

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$$\widetilde{v}_{i,p}(r) = 1 + \int_{s_{i,p}\widetilde{\mathcal{M}}_{i,p}}^{r} \widetilde{v}_{i,p}'(t)dt \ge 1 - \int_{s_{i,p}\widetilde{\mathcal{M}}_{i,p}}^{r} \frac{t}{M}dt = 1 - \frac{r^2}{2M} + \frac{(s_{i,p}\widetilde{\mathcal{M}}_{i,p})^2}{2M}.$$

Therefore by the pointwise convergence, and using (2.34) once more, we get 666

$$1 \ge \widetilde{v}(r) \ge 1 - \frac{r^2}{2M}$$

and the claim follows. 668

Since \tilde{v} is a weak solution to (2.14) that satisfies $\tilde{v}(0) = 1$ then $\tilde{v} = V_M$. Let us also 669 remark that we have proved that any sequence $p_n \rightarrow p_M$ admits a subsequence $p_{k_n} \rightarrow p_M$ 670 for which $\widetilde{v}_{i, p_{k_n}} \to V_M$, which yields that $\widetilde{v}_{i, p} \to V_M$ indeed. 671

Further $\widetilde{v}_{i,p} \to V_M$ also in $C^1(R^{-1}, R)$ for every R > 1. Indeed (2.33) and (2.35) ensure 672 that $t_{i,p} \widetilde{\mathcal{M}}_{i,p} < R^{-1} < R < t_{i+1,p} \widetilde{\mathcal{M}}_{i,p}$ for p near p_M . Therefore, remembering that 673

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674 $0 \leq \widetilde{v}_{i,p} \leq 1$, we have by (2.36)

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$$|\widetilde{v}_{i,p}'(t)| \le t^{1-M} \left| \int_{s_{i,p},\widetilde{\mathcal{M}}_{i,p}}^{t} r^{M-1} dr \right| \le C \text{ in } (R^{-1}, R) \quad ($$

thanks to (2.34). Lastly it is easy to get an uniform bound for $\tilde{v}_{i,p}''$ using the fact that $\tilde{v}_{i,p}$ is a classical solution to (2.13) in (R^{-1} , R).

It remains to prove that (2.33)–(2.35) hold true. To this aim we insert for a while the index denoting the number of nodal zones and we let then v_p^j be the nodal solution with j nodal domains. By (2.27) we have that $w_{i,p} := (t_{i,p}^m)^{\frac{2}{p-1}} v_p^m(t_{i,p}^m t)$ coincides with v_p^i on (0, 1). This implies that

$$\mathcal{M}_{i-1,p}^{i} = (t_{i,p}^{m})^{\frac{2}{p-1}} \mathcal{M}_{i-1,p}^{m}$$

684 and also that

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686 which together yields

$$t_{i,p}^{m} \widetilde{\mathcal{M}}_{i-1,p}^{m} = \widetilde{\mathcal{M}}_{i-1,p}^{i}, \qquad s_{i,p}^{m} \widetilde{\mathcal{M}}_{i,p}^{m} = s_{i,p}^{i+1} \widetilde{\mathcal{M}}_{i,p}^{i+1},$$

 $\frac{s_{i-1}^m}{t_{i,p}^m} = s_{i-1,p}^i$

689 for i = 1, ..., m - 1. Therefore (2.17) implies (2.33). We claim that

$$s_{m-1,p}^m \widetilde{\mathcal{M}}_{m-1,p}^m \to 0 \quad \text{as } p \to p_M,$$
 (2.37)

from which it follows (2.34) and then, in turn, (2.35).

For simplicity of notation we write v_p , s_p , t_p and $\widetilde{\mathcal{M}}_p$ instead of v_p^m , $s_{m-1,p}^m$, $t_{m-1,p}^m$ and $\widetilde{\mathcal{M}}_{m-1,p}^m$. We begin by checking that

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$$s_p \, \widetilde{\mathcal{M}}_p \le C. \tag{2.38}$$

⁶⁹⁵ We assume by contradiction that $s_p \widetilde{\mathcal{M}}_p \to +\infty$ and look separately at the two cases

 $\begin{array}{ll} {}^{_{696}} & (\mathrm{i}) \quad \widetilde{\mathcal{M}}_p(t_p - s_p) \to 0, \\ {}^{_{697}} & (\mathrm{ii}) \quad \widetilde{\mathcal{M}}_p(t_p - s_p) \to A \in [-\infty, 0) \end{array}$

In the first case we look at the function $\tilde{v}_p := \tilde{v}_{m-1,p}$ introduced in (2.10). It is easy to see that \tilde{v}_p is positive, increasing and concave on $(a_p, b_p) := (t_p \widetilde{\mathcal{M}}_p, s_p \widetilde{\mathcal{M}}_p)$ with $\tilde{v}_p(a_p) = 0 < \tilde{v}_p(t) < \tilde{v}_p(b_p) = 1$. So there exists a sequence $\xi_p \in (a_p, b_p)$ such that

$$\widetilde{v}'_p(\xi_p) = \frac{\widetilde{v}_p(b_p) - \widetilde{v}_p(a_p)}{b_p - a_p} = \frac{1}{b_p - a_p} \to +\infty,$$

and by concavity also $\tilde{v}'_p(a_p) \to +\infty$. On the contrary the estimate (2.21) yields

$$\widetilde{v}'_{p}(a_{p}) = \frac{1}{\mathcal{M}_{m-1,p}^{\frac{p+1}{2}}} v'_{p} \left(\frac{a_{p}}{\widetilde{\mathcal{M}}_{m-1,p}}\right) \le \frac{C_{2}t_{p}^{1-p\frac{M-2}{2}}}{(\widetilde{\mathcal{M}}_{m-1,p})^{\frac{p+1}{p-1}}} = \frac{C_{2}t_{p}^{\frac{p+1}{p-1}+1-p\frac{M-2}{2}}}{(t_{p}\widetilde{\mathcal{M}}_{m-1,p})^{\frac{p+1}{p-1}}} \to 0$$

because necessarily $t_p \widetilde{\mathcal{M}}_{m-1,p}$ diverges, since we are assuming (i), while $\frac{p+1}{p-1} + 1 - p \frac{M-2}{2}$ is positive and converges to 0.

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In the second case we introduce the notation 706

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 $\begin{aligned} A_p &= (t_p - s_p) \, \widetilde{\mathcal{M}}_p, \qquad B_p = \widetilde{\mathcal{M}}_p (1 - s_p), \\ \hat{v}_p(t) &= \frac{(-1)^{m-1}}{\mathcal{M}_p} v_p \left(\frac{t}{\widetilde{\mathcal{M}}_p} + s_p \right) \quad \text{for } t \in [A_p, B_p]. \end{aligned}$

Notice that \hat{v}_p solves 710

$$\begin{cases} -\hat{v}_{p}^{\prime\prime} - \frac{M-1}{t + \tilde{\mathcal{M}}_{p} s_{p}} \hat{v}_{p}^{\prime} = |\hat{v}_{p}|^{p-1} \hat{v}_{p} & t \in (A_{p}, B_{p}), \\ 0 < \hat{v}_{p}(t) \le \hat{v}_{p}(0) = 1, \ \hat{v}_{p}^{\prime}(0) = 0 & t \in (A_{p}, B_{p}), \\ \hat{v}_{p}(A_{p}) = 0 = \hat{v}_{p}(B_{p}), \end{cases}$$
(2.39)

with $A_p \rightarrow A < 0$ by assumption (ii) and $B_p \rightarrow +\infty$ by (2.17), (2.18). Integrating the 712 equation in (2.39) we get for $t \in [0, B_p]$ 713

714
$$\frac{|\hat{v}_{p}'(t)|}{t + \widetilde{\mathcal{M}}_{p}s_{p}} = \frac{1}{(t + \widetilde{\mathcal{M}}_{p}s_{p})^{M}} \int_{0}^{t} (\tau + \widetilde{\mathcal{M}}_{p}s_{p})^{M-1} \hat{v}_{p}^{p}(\tau) d\tau$$
715
$$\leq \frac{1}{M} \left(1 - \left(\frac{\widetilde{\mathcal{M}}_{p}s_{p}}{t + \widetilde{\mathcal{M}}_{p}s_{p}} \right)^{M} \right) \leq \frac{1}{M}.$$

Besides taking $t \in [-\delta, 0]$ with $0 < \delta < -A/2$ and integrating the equation in (2.39) on 717 (t, 0) gives 718

$$\frac{|\hat{v}'_p(t)|}{t + \widetilde{\mathcal{M}}_p s_p} = \frac{1}{(t + \widetilde{\mathcal{M}}_p s_p)^M} \int_t^0 (\tau + \widetilde{\mathcal{M}}_p s_p)^{M-1} \hat{v}_p^p(\tau) d\tau$$

$$\leq \frac{1}{M} \left(\left(\frac{\widetilde{\mathcal{M}}_p s_p}{t + \widetilde{\mathcal{M}}_p s_p} \right)^M - 1 \right) \leq \frac{1}{M} \left(\left(\frac{\widetilde{\mathcal{M}}_p s_p}{-\delta + \widetilde{\mathcal{M}}_p s_p} \right)^M - 1 \right) \leq C(\delta).$$

721

So \hat{v}_p converges in $C^1[0, +\infty)$ to a bounded weak solution of 722

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which is non-trivial because $\hat{v}(0) = 1$. This is not possible because \hat{v} should be strictly 724 convex. 725

 $-\hat{v}'' = \hat{v}^{p_M}$

Now that it has been assured that $s_p \mathcal{M}_p$ is at least bounded, we take that (2.37) does not 726 hold, which means that (up to a subsequence) $s_p \widetilde{\mathcal{M}}_p \to s_0 > 0$. We check that it is not 727 possible by arguing separately according whether 728

(I) $t_p \widetilde{\mathcal{M}}_p \to s_0,$ (II) $t_p \widetilde{\mathcal{M}}_p \to 0,$ 729

730

731 (III) or
$$t_p \widetilde{\mathcal{M}}_p \to t_0 \in (0, s_0)$$
.

Case (I) can be ruled out arguing as in the previous case i). Also here we get that $\widetilde{v}'_{p}(t_{p},\widetilde{\mathcal{M}}_{p}) \rightarrow$ 732 $+\infty$, while estimate (2.21) would imply that it stays bounded. 733

Otherwise in case (II) we consider again the function $\tilde{v}_p := \tilde{v}_{m-1,p}$ introduced in (2.10) and 734 extended to zero outside $(t_p \widetilde{\mathcal{M}}_p, \widetilde{\mathcal{M}}_p)$ so that it belongs to $\mathcal{D}_M(0, \infty)$ and by Corollary 2.6 is 735 uniformly bounded in $\mathcal{D}_M(0,\infty)$. Now $(t_p\mathcal{M}_p,\mathcal{M}_p)$ invades $(0,\infty)$ because we are taking 736 737

that $t_p \mathcal{M}_p \to 0$ and (2.17) holds. Then the same arguments used in the first part of the proof

show that $\tilde{v}_p \to \tilde{v}$ weakly in $\mathcal{D}_M(0,\infty)$ and in $C^1_{\text{loc}}(0,\infty)$, where \tilde{v} weakly solves 738

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$$-\left(t^{M-1}\tilde{v}'\right)' = t^{M-1}\tilde{v}^{p_M}, \quad \text{as } t > 0.$$
(2.40)

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Asymptotic profile and Morse index of nodal radial solutions to...

Therefore \tilde{v} has to be a suitable rescaling of the function V_M , as showed in the "Appendix". 740 In particular it has only one critical point at r = 0. On the other hand the functions \tilde{v}_p have a 741 critical point at $s_p \widetilde{\mathcal{M}}_p \to s_0 > 0$, and by the convergence in $C^1_{\text{loc}}(0, \infty) s_0$ is a critical point 742 for \tilde{v} . 743

At last case (III) can be ruled out following the line of case b) in the proof of Proposition 2.2. 744

Precisely we look at the function $w_p = w_{m-1,p}$ introduced in (2.27), and check the hypothe-745

sets of Lemma 2.10. Equation (2.30), i.e. $\rho_p = t_{m-1,p}^m \widetilde{\mathcal{M}}_{m-2,p}^m \to +\infty$ is ensured by (2.33). 746 Concerning (2.29), it is trivial for m = 2, while for $m \ge 3$ rescaling we get 747

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$$\widetilde{\mathcal{M}}_{m-2,p}^{m}\widetilde{\mathcal{M}}_{m-2,p}^{m} = s_{m-2,p}^{m-1}\widetilde{\mathcal{M}}_{m-2,p}^{m-1} \leq C$$

by the previously proved property (2.38), so that 749

s

$$\tau_p = s_{m-2,p}^m / t_{m-1,p}^m \le C / t_{m-1,p}^m \widetilde{\mathcal{M}}_{m-2,p}^m = C / \rho_p \to 0.$$

So Lemma 2.10 gives that $w_p \to 0$ uniformly on any set of type $[1 - \delta, 1]$ with $0 < \delta < 1$. 751 In particular it is uniformly bounded on $[1 - \delta, 1]$. On the other hand w_p is bounded also in 752

 $[1, 1/t_p]$ (uniformly w.r.t. p) because 753

754

$$|w_p(r)| \le t_p^{\frac{2}{p-1}} \mathcal{M}_p = \left(t_p \widetilde{\mathcal{M}}_p\right)^{\frac{2}{p-1}} \le C$$

by assumption. Moreover s_p/t_p is a critical point for w_p which converges to s_0/t_0 , and the 755 corresponding maximum value is 756

$$w_p(s_p/t_p) = t_p^{\frac{2}{p-1}} |v_p(s_p)| = (t_p \widetilde{\mathcal{M}}_p)^{\frac{2}{p-1}} \to t_0^{\frac{M-2}{2}}$$

Integrating the equation in (2.28) gives 758

$$|w'_{p}(r)| \le r^{1-M} \int_{\frac{s_{p}}{t_{p}}}^{r} t^{M-1} |w_{p}(t)|^{p} dt \le C$$

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whenever $r \in (1 - \delta, R)$ for any fixed R > 1. Next since w_p is a classical solution to (2.28) 761 it is easily seen that also $|w_p''(r)|$ is bounded for $r \in (1 - \delta, R)$, so that w_p converges in 762 $C_{\rm loc}^1(1-\delta,+\infty)$ to a function w that weakly satisfies 763

$$\begin{cases} -\left(t^{M-1}w'\right)' = t^{M-1}|w|^{p_M-1}w & \text{as } t > 1-\delta, \\ w(s_0/t_0) = t_0^{\frac{M-2}{2}} > 0, \\ w(1) = 0. \end{cases}$$

This is not possible because w should be identically zero by the unique continuation principle, 765 as we have seen that w coincides with zero on $(1 - \delta, 1]$. 766

2.2 Some consequences of the convergence result 767

We conclude this section by pointing out some qualitative properties of the auxiliary functions 768

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$$z_p(r) = rv'_p(r) + \frac{2}{p-1}v_p(r)$$
 for $0 \le r < 1$, (2.41)

$$f_p(r) = pr^2 |v_p(r)|^{p-1}$$
 for $0 \le r < 1$, (2.42)

$$\tilde{f}_{i,p}(r) = f_p\left(\frac{r}{\widetilde{\mathcal{M}}_{i,p}}\right) = pr^2 |\tilde{v}_{i,p}(r)|^{p-1} \quad \text{for } t_{i,p}\widetilde{\mathcal{M}}_{i,p} < r < t_{i+1,p}\widetilde{\mathcal{M}}_{i,p}, \quad (2.43)$$

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(for i = 0, ..., m - 1) that can be deduced by the convergence established in Propositions 2.2, 2.3 and 2.9, and shall be useful when investigating the asymptotic behavior of the eigenfunctions and eigenvalues related to v_p , in next section.

Lemma 2.11 The function z_p has exactly m zeros in (0, 1), one in each nodal domain ($t_{i,p}, t_{i+1,p}$) of v_p , that we denote by $\xi_{i,p}$ for i = 0, 1, ..., m - 1.

Moreover $\xi_{i,p}$ is the unique critical point in the nodal domain $(t_{i,p}, t_{i+1,p})$ of the function f_p , which is strictly increasing in $(t_{i,p}, \xi_{i,p})$ and strictly decreasing in $(\xi_{i,p}, t_{i+1,p})$. Further $s_{i,p} < \xi_{i,p} < t_{i+1,p}$.

Here we denote $t_{0,p} = 0$ and $t_{m,p} = 1$.

Proof The first part of the statement, concerning z_p , has been proved in [5, Lemma 4.7]. Next it suffices to compute

$$f'_{p} = (p-1)r|v_{p}|^{p-3}v_{p}\left(\frac{2}{p-1}v_{p} + rv_{p}'\right) = (p-1)r|v_{p}|^{p-3}v_{p}z_{p},$$

as $r \neq t_{i,p}$, and the second part of the statement follows trivially. In particular $\xi_{i,p} > s_{i,p}$ because in the subset $(t_{i,p}, s_{i,p})$ the functions v_p and v'_p have the same sign, so that $f'_p > 0$.

Lemma 2.12 For every $i = 0, \ldots m - 1$, as $p \rightarrow p_M$ we have

$$\tilde{f}_{i,p}(r) \to F(r) = \frac{(M+2)r^2}{M-2} \left(1 + \frac{r^2}{M(M-2)}\right)^{-2}$$
 (2.44)

⁷⁹¹ uniformly in $[R^{-1}, R]$ for every R > 1 and also in [0, R] when i = 0. Moreover

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$$_{i,p}\widetilde{\mathcal{M}}_{i,p} \to \bar{\xi} \in (0,\infty)$$

$$(2.45)$$

⁷⁹⁴ where $\overline{\xi}$ is the unique maximum point of the function *F*.

Proof The convergence of $\tilde{f}_{i,p}$ is an immediate consequence of the one of $\tilde{v}_{i,p}$ stated in Propositions 2.9 and 2.3. Notice that while proving Proposition 2.3 we have shown that $t_{i,p}\widetilde{\mathcal{M}}_{i,p} \to 0$ and $t_{i+1,p}\widetilde{\mathcal{M}}_{i,p} \to +\infty$. Since the function *F* has only one critical point $\bar{\xi} \in (0, +\infty)$, which is its maximum point, it follows that the maximum point of $\tilde{f}_{i,p}$ converges to $\bar{\xi}$. On the other hand it is clear by construction that the maximum point of $\tilde{f}_{i,p}$ is $\xi_{i,p}\widetilde{\mathcal{M}}_{i,p}$.

Let us also recall an estimate obtained in [18, Proposition 3.6] for integer values of Mthat we extend to every value of M.

Lemma 2.13 The function f_p satisfies $0 \le f_p(r) \le C$ for $r \in [0, 1]$, uniformly w.r.t. p in a left neighborhood of p_M .

⁸⁰⁵ We report here a slightly different proof, in view of further estimates that we aim to obtain.

Proof The first assertion of Lemma 2.7 implies that for every $r \in [0, t_{1,p})$

$$0 \le f_p(r) \le p g_p(\widetilde{\mathcal{M}}_{0,p}r)$$
 being $g_p(s) := \frac{s^2}{(1+s^2)^{\frac{(M-2)(p-1)}{2}}}$

Since the functions g_p are uniformly bounded on $[0, +\infty)$ (as $p \ge \frac{M}{M-2}$), it follows that also f_p are uniformly bounded on $[0, t_{1,p}]$.

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(2.46)

Next we know that, for every i = 1, ..., m-1 and $K > 0, \tilde{v}_{i,p} \to V_M$ uniformly in $[\frac{1}{K}, K]$. As V_M has a positive minimum on the set $[\frac{1}{K}, K]$, it follows that

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$$|\widetilde{v}_{i,p}(r)| \le 2 V_M(r) \quad \text{in} \left[\frac{1}{\kappa}, K\right]$$

as $p_M - \delta for some <math>\delta = \delta(K) > 0$. As in the previous step it follows that

$$f_p(r) \le 2p g_{p_M}(\widetilde{\mathcal{M}}_{i,p}r) \le C$$

in the interval $[(K\widetilde{\mathcal{M}}_{i,p})^{-1}, K\widetilde{\mathcal{M}}_{i,p}^{-1}]$ for $p \in (p_M - \delta, p_M)$.

On the other hand in force of (2.45) we can choose the parameter K in such a way that the maximum point of $f_p(r)$ in the interval $(t_{i,p}, t_{i+1,p})$, i.e. $\xi_{i,p}$, is contained in $[(K\widetilde{\mathcal{M}}_{i,p})^{-1}, K\widetilde{\mathcal{M}}_{i,p}^{-1}]$, implying that $0 \leq f_p(r) \leq C$ in the interval $(t_{i,p}, t_{i+1,p})$ for $i = 1, \ldots, m-1$ concluding the proof.

Similar arguments also allow us to show the following estimate.

Lemma 2.14 For every $\varepsilon > 0$ there exist $\overline{K} = \overline{K}(\varepsilon) > 0$ and $\overline{p} = \overline{p}(\varepsilon, \overline{K}) > 0$ such that, denoting by

$$G_{i,p}(K) := \{r \in (0,1) : K(\widetilde{\mathcal{M}}_{i-1,p})^{-1} < r < (K\widetilde{\mathcal{M}}_{i,p})^{-1}\} \quad \text{for } i = 1, \dots, m-1,$$

$$g_{25}^{825} \qquad G_{m,p}(K) := \{ r \in (0,1) : K(\mathcal{M}_{m-1,p})^{-1} < r < 1 \}$$

827 it holds

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$$\max\left\{f_p(r): r \in \bigcup_{i=1}^m G_{i,p}(K)\right\} < \varepsilon$$
(2.47)

for any $K > \overline{K}$ provided that $p \in (\overline{p}, p_M)$.

Proof To begin with we choose $\bar{K} > 0$ such that $K > \max{\{\bar{\xi}, \bar{\xi}^{-1}\}}$ and $p_M g_{p_M}(K^{-1})$, $p_M g_{p_M}(K) < \varepsilon/2$ for any $K > \bar{K}$. Here $\bar{\xi}$ is the maximum point of the function F mentioned in Lemma 2.12 and g_{p_M} is the same function introduced in the proof of Lemma 2.13, and the choice of \bar{K} is possible because $g_{p_M}(0) = 0 = \lim_{r \to +\infty} g_{p_M}(r)$.

Next (2.46) yields that there exists $p_1 = p_1(\vec{K}, \varepsilon)$ such that

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$$f_p((\bar{K}\widetilde{\mathcal{M}}_{i,p})^{-1}), \ f_p(\bar{K}\widetilde{\mathcal{M}}_{i,p}^{-1}) < \varepsilon \text{ for } p_1 < p < p_M \text{ and } i = 0, \dots m - 1.$$

Then (2.45) yields that there exists $\bar{p} = \bar{p}(\bar{K}) > p_1$ such that $\xi_{i,p}$, the unique critical point of f_p in the interval $(t_{i,p}, t_{i+1,p})$, satisfies

$$(K\widetilde{\mathcal{M}}_{i,p})^{-1} < (\bar{K}\widetilde{\mathcal{M}}_{i,p})^{-1} < \xi_{i,p} < \bar{K}\widetilde{\mathcal{M}}_{i,p}^{-1} < K\widetilde{\mathcal{M}}_{i,p}^{-1} \quad \text{for } \bar{p} < p < p_M$$

and i = 0, ..., m - 1, for any $K > \overline{K}$. Remembering also that f_p is increasing in $(t_{i,p}, \xi_{i,p})$ and decreasing in $(\xi_{i,p}, t_{i+1,p})$ by Lemma 2.11, it follows that

$$f_p(r) \le f_p\left(\left(K\widetilde{\mathcal{M}}_{i,p}\right)^{-1}\right) < \varepsilon \quad \text{for } K(\widetilde{\mathcal{M}}_{i,p})^{-1} < r < t_{i+1,p}, \quad \text{for } i = 0, \dots, m-1,$$

$$f_p(r) \le f_p\left(K\widetilde{\mathcal{M}}_{i,p}^{-1}\right) < \varepsilon \quad \quad \text{for } t_{i,p} < r < (K\widetilde{\mathcal{M}}_{i,p})^{-1}, \quad \text{for } i = 1, \dots, m-1$$

for any $K > \overline{K}$, for the same values of p.

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3 The computation of the Morse index

In this section we address the computation of the Morse index of the nodal radial solution u_p of (1.1) when *p* approaches the threshold p_{α} . By definition the Morse index of u_p , that we denote by $m(u_p)$, is the maximal dimension of a subspace of $H_0^1(B)$ in which the quadratic form

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$$\mathcal{Q}_p(w) := \int_B \left(|\nabla w|^2 - p \, |x|^\alpha |u_p|^{p-1} w^2 \right) dx$$

is negative definite, or equivalently, is the number, counted with multiplicity, of the negative eigenvalues in $H_0^1(B)$ of

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$$\begin{cases} -\Delta \phi - p |x|^{\alpha} |u_p|^{p-1} \phi = \Lambda_i(p) \phi & \text{in } B\\ \phi = 0 & \text{on } \partial B. \end{cases}$$
(3.1)

Similarly the radial Morse index of u_p , denoted by $m_{rad}(u_p)$, is the number of negative eigenvalues of (3.1) in $H^1_{0,rad}(B)$, namely the eigenvalues of (3.1) associated with a radial eigenfunction. It has been proved in [3, Proposition 1.1] (since $p|x|^{\alpha}|u_p|^{p-1} \in L^{\infty}(B)$) that the number of negative eigenvalues of (3.1) in $H^1_0(B)$ (or in $H^1_{0,rad}(B)$), counted with multiplicity, coincides with the number of negative eigenvalues of the singular eigenvalue problem

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$$\begin{cases} -\Delta\widehat{\phi} - p|x|^{\alpha}|u_p|^{p-1}\widehat{\phi} = \frac{\widehat{\Lambda}_i(p)}{|x|^2}\widehat{\phi} & \text{in } B \setminus \{0\}\\ \widehat{\phi} = 0 & \text{on } \partial B, \end{cases}$$
(3.2)

in $H_0^1(B)$ (or in $H_{0,\text{rad}}^1(B)$). This allows us to give this alternative definition of Morse index:

Definition 3.1 (Alternative definition of Morse index) The Morse index of u_p is the number, counted with multiplicity of the negative singular eigenvalues $\widehat{\Lambda}_i(p)$ of (3.2) in $H_0^1(B)$. Moreover the radial Morse index of u_p is the number of negative singular radial eigenvalues $\widehat{\Lambda}_i^{\text{rad}}(p)$ of (3.2) in $H_{0,\text{rad}}^1(B)$.

These eigenvalues $\widehat{\Lambda}_i(p)$ are well defined in $H_0^1(B)$ (by the Hardy inequality) as far as $\widehat{\Lambda}_i(p) < \left(\frac{N-2}{2}\right)^2$ and have the useful property that can be decomposed as

$$\widehat{\Lambda}_i(p) = \widehat{\Lambda}_k^{\rm rad}(p) + \lambda_j, \qquad (3.3)$$

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where
$$\lambda_j = j(N + j - 2)$$
 are the eigenvalues of the Laplace–Beltrami operator on the
sphere \mathbb{S}_{N-1} , and $\widehat{\Lambda}_k^{\text{rad}}(p)$ are the radial singular eigenvalues of (3.2) which are all simple,
see [3] where a complete study of the singular eigenvalues and their properties has been done.
Further if $\widehat{\phi}$ is a radial eigenfunction of (3.2), the function

$$\psi(t) = \widehat{\phi}(r)$$
 with $t = r^{\frac{2+\alpha}{2}}$

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$$\begin{cases} -\left(t^{M-1}\psi'\right)' - t^{M-1}p|v_p|^{p-1}\psi = t^{M-3}\widehat{v}_i(p)\psi \text{ for } t \in (0,1)\\ \psi \in H^1_{0,M} \end{cases}$$
(3.4)

where v_p as in (2.2) is a solution to (2.3) as in Sect. 2 and $M = M(\alpha, N)$ has been defined in (2.4). These eigenvalues $\hat{v}_i(p)$ are well defined in $H_{0,M}^1$ as far as $\hat{v}_i(p) < \left(\frac{M-2}{2}\right)^2$ and

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$$\widehat{\Lambda}_{i}^{\mathrm{rad}}(p) = \left(\frac{2+\alpha}{2}\right)^{2} \widehat{\nu}_{i}(p).$$
(3.5)

To deal with problem (3.4) we define by \mathcal{L}_M the Lebesgue space

$$\mathcal{L}_M := \{ w : (0, 1) \to \mathbb{R} \text{ measurable and s.t. } \int_0^1 t^{M-3} w^2 dt < +\infty \}$$

with the scalar product $\int_0^1 r^{M-3} \psi w \, dr$, which gives the orthogonality condition

$$w \perp_M \psi \iff \int_0^1 t^{M-3} w \psi dt = 0 \text{ for } w, \psi \in \mathcal{L}_M.$$

In virtue of an extended radial Hardy inequality for $H_{0,M}^1$ in [3, Lemma 5.5] $H_{0,M}^1 \subset \mathcal{L}_M$ and this allows us to characterize the eigenvalues $\hat{\nu}$ by the minimization problems

$$\widehat{\nu}_{1}(p) = \inf_{\substack{w \in H_{0,M}^{1} \\ w \neq 0}} \frac{\int_{0}^{1} t^{M-1} \left((w')^{2} - p |v_{p}|^{p-1} w^{2} dt \right) dr}{\int_{0}^{1} t^{M-3} w^{2} dt},$$

$$\widehat{\nu}_{i}(p) = \inf_{\substack{w \in H_{0,M}^{1} \\ w \neq 0 \\ w \perp_{M} \{\psi_{1}, \dots, \psi_{i-1}\}}} \frac{\int_{0}^{1} t^{M-1} \left((w')^{2} - p |v_{p}|^{p-1} w^{2} dt \right) dr}{\int_{0}^{1} t^{M-3} w^{2} dr}$$
(3.6)

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where
$$\psi_j$$
 for $j = 1, ..., m - 1$ denotes an eigenfunction associated with $\hat{\nu}_j$. Every time $\hat{\nu}_i < \left(\frac{M-2}{2}\right)^2$, the function which attains $\hat{\nu}_i$ is a weak solution to (3.4) meaning that

$$\int_{0}^{1} t^{M-1} \psi' \varphi' \, dt - p \int_{0}^{1} t^{M-1} |v_{p}|^{p-1} \psi \varphi \, dt = \widehat{\nu}_{i}(p) \int_{0}^{1} t^{M-3} \psi \varphi \, dt \qquad (3.7)$$

for every $\varphi \in H_{0,M}^1$. These generalized radial singular eigenvalues $\hat{v}_i(p)$, (associated with v_p) have been studied in [3, Sect. 3.1] where it is proved that they are all *simple* and that eigenfunctions associated with different eigenvalues are orthogonal in \mathcal{L}_M . Moreover the only negative eigenvalues of (3.4) are

$$\widehat{\nu}_1(p) < \widehat{\nu}_2(p) < \dots < \widehat{\nu}_m(p) < 0 \tag{3.8}$$

894 and satisfy

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$$\widehat{\nu}_i(p) < -(M-1)$$
 for $i = 1, \dots m-1$, (3.9)

$$-(M-1) < \hat{\nu}_m(p) < 0, \tag{3.10}$$

for any value of the parameter p, see [5, Proposition 3.3 and Theorem 1.3]. Then (3.5), together with Definition 3.1, implies that $m_{rad}(u_p) = m$, the number of the nodal zones of u_p .

⁹⁰¹ Furthermore putting together Proposition 1.4 of [3] and Theorem 1.3 from [5] we have

Proposition 3.2 Let $\alpha \ge 0$ and let u_p be any radial solution to (1.1) with m nodal zones. The Morse index of u_p is given by

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 $m(u_p) = \sum_{i=1}^{m} \sum_{j=0}^{\lceil J_i - 1 \rceil} N_j$ (3.11)

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where $\lceil t \rceil = \min\{k \in \mathbb{Z} : k \ge t\}$ stands for the ceiling function,

$$J_i(p) = \frac{2+\alpha}{2} \left(\sqrt{\left(\frac{M-2}{2}\right)^2 - \widehat{\nu}_i(p) - \frac{M-2}{2}} \right),$$

909 and

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$$N_j = \frac{(N+2j-2)(N+j-3)}{(N-2)!\,j!}$$

stands for the multiplicity of the eigenvalue $\lambda_j = j(N + j - 2)$ of the Laplace–Beltrami operator in the sphere \mathbb{S}_{N-1} .

Therefore the asymptotic Morse index of u_p as $p \to p_{\alpha}$ can be deduced, by the asymptotic behavior of the generalized radial singular eigenvalues $\hat{v}_i(p)$ and of the related eigenfunctions $\psi_{i,p}$ of (3.4) as $p \to p_M$ which are associated with the function v_p defined in (2.2) and studied in Sect. 2. This will be the topic of the remaining of this section.

918 3.1 Asymptotics of the singular eigenvalues $\hat{v}_i(p)$ for $i = 1, \dots, m - 1$

For simplicity of notation in the present subsection and in the next one we shall write $v_j(p)$ instead of $\hat{v}_j(p)$, and we will denote by $\psi_{j,p} \in H^1_{0,M}$ the corresponding eigenfunction to (3.4) normalized such that

$$\int_{0}^{1} r^{M-3} \psi_{j,p} \psi_{k,p} dr = \delta_{jk}.$$
(3.12)

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For every $i = 0, \dots, m-1$ and $j = 1, \dots, m$ we also introduce the rescaled eigenfunctions

$$\widetilde{\psi}_{j,p}^{i}(r) := \begin{cases} (\widetilde{\mathcal{M}}_{i,p})^{\frac{2-M}{2}} \psi_{j,p}\left(\frac{r}{\widetilde{\mathcal{M}}_{i,p}}\right) & \text{if } \widetilde{\mathcal{M}}_{i,p}t_{i,p} < r < \widetilde{\mathcal{M}}_{i,p}t_{i+1,p}, \\ 0 & \text{elsewhere,} \end{cases}$$
(3.13)

where $t_{i,p}$, $t_{i+1,p}$ are the zeros of v_p as in Sect. 2 and $\widetilde{\mathcal{M}}_{i,p}$ is as in (2.11), in such a way that

$$\int_{0}^{\infty} r^{M-3} (\tilde{\psi}_{j,p}^{i})^{2} dr \leq \int_{0}^{1} r^{M-3} \psi_{j,p}^{2} dr = 1, \qquad (3.14)$$

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$$\int_0^\infty r^{M-1} ((\widetilde{\psi}_{j,p}^i)')^2 \, dr \le \int_0^1 r^{M-1} (\psi_{j,p}')^2 \, dr. \tag{3.15}$$

Then the functions $\tilde{\psi}_{j,p}^i$ belong to the space $\mathcal{D}_M(0,\infty)$ for every $i = 0, \ldots, m-1$ and $j = 1, \ldots, m$ since $\psi_j \in H^1_{0,M}$ and they satisfy

$$(932 - \left(r^{M-1}(\widetilde{\psi}_{j,p}^{i})'\right)' = r^{M-1}\left(W_{p}^{i} + \frac{v_{j}(p)}{r^{2}}\right)\widetilde{\psi}_{j,p}^{i} \text{ as } \widetilde{\mathcal{M}}_{i,p}t_{i,p} < r < \widetilde{\mathcal{M}}_{i,p}t_{i+1,p}$$

933 where

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$$W_{p}^{i}(r) = p |\widetilde{v}_{i,p}(r)|^{p-1}$$
 (3.17)

and $\tilde{v}_{i,p}$ is as defined in (2.10). By the asymptotics of $\tilde{v}_{i,p}$ in Propositions 2.3 and 2.9 we have that

 $W_p^i(r) \to W(r) = \frac{M+2}{M-2} \left(1 + \frac{r^2}{M(M-2)}\right)^{-2}$ (3.18)

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

⁹³⁸ in $C_{\text{loc}}^1[0,\infty)$ for i = 0 and in $C_{\text{loc}}^1(0,\infty)$ for $i = 1,\ldots,m-1$, therefore the eigenvalue ⁹³⁹ problems (3.16) have a unique limit problem which is the following

$$-\left(r^{M-1}(\widetilde{\psi})'\right)' = r^{M-1}\left(W + \frac{\beta}{r^2}\right)\widetilde{\psi} \text{ as } r \in (0,\infty),$$
(3.19)

and admits as nonpositive eigenvalues in the space $\mathcal{D}_M(0, \infty)$ only the two values $\beta_1 = -(M-1)$ and $\beta_2 = 0$ with corresponding eigenfunctions

$$_{1}(r) = \frac{r}{\left(1 + \frac{r^{2}}{M(M-2)}\right)^{\frac{M}{2}}}, \qquad \eta_{2}(r) = \frac{1 - \frac{r^{2}}{M(M-2)}}{\left(1 + \frac{r^{2}}{M(M-2)}\right)^{\frac{M}{2}}}$$
(3.20)

see the "Appendix". We recall that an eigenfunction η is a weak solution to (3.19) if it satisfies

$$\int_0^\infty r^{M-1} \eta' \varphi' \, dr = \int_0^\infty r^{M-1} \left(W + \frac{\beta}{r^2} \right) \eta \varphi \, dr \tag{3.21}$$

947 for every $\varphi \in \mathcal{D}_M(0,\infty)$.

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Let us prove some useful properties, which inherit all the *m* negative eigenvalues and the related eigenfunctions.

Lemma 3.3 There exist $\delta > 0$ and C > 0 such that for every $p \in (p_M - \delta, p_M)$ we have

$$-C \le v_1(p) < v_2(p) \dots < v_m(p) < 0$$
(3.22)

$$\int_{0}^{\infty} r^{M-1} ((\tilde{\psi}_{j,p}^{i})')^{2} dr \le C$$
(3.23)

954 for every i = 0, ..., m - 1 and j = 1, ...m.

Proof Using $\psi_{j,p}$ as a test function in (3.7) gives

$$\int_{0}^{1} r^{M-1} \left(\psi_{j,p}^{\prime} \right)^{2} = \int_{0}^{1} r^{M-1} \left(p |v_{p}|^{p-1} + \frac{v_{j}(p)}{r^{2}} \right) \psi_{j,p}^{2} dr$$

$$= \int_{0}^{1} r^{M-3} \left(f_{p} + v_{j}(p) \right) \psi_{j,p}^{2} dr$$
(3.24)

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where f_p is as defined in (2.42). Taking advantage from (3.12) one can extract $v_1(p)$ getting that

$$\nu_{1}(p) = \int_{0}^{1} r^{M-1} \left(\psi_{1,p}' \right)^{2} - r^{M-3} f_{p} \psi_{1,p}^{2} dr \ge -\sup_{r \in (0,1)} f_{p}(r) \int_{0}^{1} r^{M-3} \psi_{1,p}^{2} dr = -C$$

for *p* near p_M , thanks to Lemma 2.13.

Besides, since $v_j(p) < 0$ for j = 1, ..., m by (3.8), (3.24) also yields that

$$\int_0^1 r^{M-1} \left(\psi'_{j,p} \right)^2 < \int_0^1 r^{M-3} f_p \psi_{j,p}^2 \, dr \le \sup_{r \in (0,1)} f_p(r) \int_0^1 r^{M-3} \psi_{j,p}^2 \, dr = C.$$

So also (3.23) is proved, recalling (3.15).

From the boundedness of the eigenfunctions in (3.23) it is easy to deduce that they converge to eigenfunctions of the limit problem (3.19).

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 $v_i(p_n) \to \bar{v}_i$

Lemma 3.4 Let j = 1, ...m and p_n be a sequence in $(1, p_M)$ with $p_n \to p_M$. Then there exist a subsequence (that we still denote by p_n), a number $\bar{v}_j \leq 0$, a weak solution to (3.19) with $\beta = \bar{v}_j$, called η , and m numbers $A_j^0, ..., A_j^{m-1} \in \mathbb{R}$ such that

$$\widetilde{\psi}^{i}_{j,p_{n}} \to A^{i}_{j}\eta \quad weakly in \mathcal{D}_{M}(0,\infty) \text{ and strongly in } C^{1}_{\text{loc}}(0,\infty)$$

for i = 0, ..., m - 1. Further for j = 1, ..., m - 1 the sequence $\tilde{\psi}_{j,p_n}^0$ converges to $A_j^0 \eta$ also in $C_{\text{loc}}^1[0,\infty)$.

Proof By (3.9), (3.10) and (3.22) it is clear that there is a subsequence $v_{j,p_n} \rightarrow \bar{v}_j \leq 0$. 975 Moreover the normalization (3.14) and the estimate (3.23) imply that $\tilde{\psi}_{i,p}^{i}$ are uniformly 976 bounded in $\mathcal{D}_M(0,\infty)$ for $i=0,\ldots,m-1$. Then, up to another subsequence $\widetilde{\psi}_{i,p_n}^i$ converges 977 to a function η weakly in $\mathcal{D}_M(0,\infty)$. It is not hard to see that one can pass to the limit in 978 the weak formulation of (3.16), getting that η is a weak solution to (3.19) with $\beta = \bar{\nu}_i \leq 0$. 979 Indeed (2.33) and (2.35) ensure that, for every $\varphi \in C_0^{\infty}(0, \infty)$ and for *n* sufficiently large, 980 the support of φ is contained in $(t_{i,p_n} \widetilde{\mathcal{M}}_{i,p_n}, t_{i+1,p_n} \widetilde{\mathcal{M}}_{i,p_n})$, where (3.16) holds. Moreover $\widetilde{\psi}_{j,p_n}^i$ converges to η also in $L^2_M(R^{-1}, R)$ as well as in $\mathcal{L}_M(R^{-1}, R)$ for every R > 1, by [3, 981 982 Lemma 5.4]. 983

Besides $\eta \in \mathcal{D}_M(0, \infty)$ and hence $\eta \in H^1_M(0, R)$ for every R > 0, and by [12, VIII.2] $\eta \in C^1(0, R)$. If $r_1, r_2 > R^{-1} > 0$ we have

$$\begin{aligned} \left| \widetilde{\psi}_{j,p}^{i}(r_{1}) - \widetilde{\psi}_{j,p}^{i}(r_{2}) \right| &\leq \int_{r_{1}}^{r_{2}} |(\widetilde{\psi}_{j,p}^{i})'(t)| dt \leq \int_{\text{Holder and (3.23)}} C \left(\int_{r_{1}}^{r_{2}} t^{1-M} dt \right)^{\frac{1}{2}} \\ &\leq C R^{\frac{M-1}{2}} \sqrt{|r_{1} - r_{2}|}, \end{aligned}$$

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so the Ascoli Theorem ensures that (up to another subsequence) $\tilde{\psi}_{j,p_n}^i \to \eta$ uniformly in any set of type $[R^{-1}, R]$. Next taking advantage from the equation in (3.16) it is easy to get a bound for $\tilde{\psi}_{j,p}^i$ in $C^2(R^{-1}, R)$ which ensures that it actually converges in $C^1(R^{-1}, R)$.

⁹⁹² Further when i = 0 we also know that $W_{p_n}^0$ is uniformly convergent (and therefore ⁹⁹³ uniformly bounded) on any set of type [0, *R*]. Consequently the arguments in [16, Lemma ⁹⁹⁴ 5.9] and [3, Proposition 3.8] prove that

$$\left| (\widetilde{\psi}_{j,p_n}^0)'(r) \right| \le Cr^{\theta_j(p_n)-1}, \quad \theta_j(p_n) = \sqrt{\left(\frac{M-2}{2}\right)^2 - \nu_j(p_n)} - \frac{M-2}{2} \quad (3.25)$$

on [0, R]. Moreover when j = 1, ..., m - 1 the estimate (3.9) ensures that $\theta_j(p_n) > 1$ for every *n*. Therefore (3.25) states that $(\tilde{\psi}_{j,p_n}^0)'$ is uniformly bounded also in [0, R], and Ascoli Theorem gives uniform convergence of $\tilde{\psi}_{j,p_n}^0$ in [0, R] as before. The C^1 convergence then follows from the uniform converge of $\tilde{\psi}_{j,p_n}^0$ recalling that integrating (3.16) and using (3.25) one easily gets

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 $(\widetilde{\psi}_{j,p_n}^0)' = -r^{1-M} \int_0^r t^{M-1} \left(W_{p_n}^0 + \frac{\nu_j(p_n)}{t^2} \right) \widetilde{\psi}_{j,p_n}^0 dt.$

Remark 3.5 Since the eigenvalues and eigenfunctions of the limit problem (3.19) are known, an immediate consequence of Lemma 3.4 is that either $A_j^i = 0$ for every i = 0, ..., m-1, or $\bar{\nu}_i$ takes one of the values -(M-1) and 0, and either $\eta = \eta_1$ (if $\bar{\nu}_i = -(M-1)$) or $\eta = \eta_2$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

(if $\bar{\nu}_i = 0$). Further when $j = 1, \dots, m-1$ the inequality (3.9) ensures that $\bar{\nu}_i = -(M-1)$ 1006 and therefore $\eta = \eta_1$. Concerning j = m, the corresponding inequality (3.10) leaves open 1007 also the possibility $\bar{\nu}_m = 0$ and $\eta = \eta_2$. 1008

The previous remark puts in evidence that the eigenvalue v_m has to be treated separately. We deal by now with the first m-1 eigenvalues and show that 1010

Proposition 3.6 Let $i = 1, \ldots, m-1$, then 1011

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$$\lim_{p \to p_M} \nu_j(p) = -(M-1).$$

Moreover for any sequence p_n in $(1, p_M)$ with $p_n \rightarrow p_M$ there exist a subsequence (that we 1013 still denote by p_n), and m numbers $A_j^0, \ldots A_j^{m-1} \in \mathbb{R}$ not simultaneously null such that 1014

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 $\widetilde{\psi}^{i}_{i} \ p_{n} \rightarrow A^{i}_{i} \eta_{1}$

for every i = 0, ..., m - 1, weakly in $\mathcal{D}_M(0, \infty)$, and strongly in $C^1_{loc}(0, \infty)$, and also in 1016 $C_{loc}^{1}[0,\infty)$ for i = 0. 1017

Proof As mentioned in Remark 3.5, it suffices to rule out the possibility that for every 1018 $i=0,\ldots,m-1,$ 1019

$$\widetilde{\psi}_{j,p}^i \to 0$$
 uniformly in any set $[R^{-1}, R]$ and also in $[0, R]$ if $i = 0.$ (3.26)

We show here that if (3.26) holds true then 1021

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$$\int_0^1 r^{M-3} f_p \psi_{j,p}^2 \, dr \to 0, \tag{3.27}$$

where f_p is as in (2.42). This is not possible (and so the proof is completed) because repeating 1023 the computations in the proof of Lemma 3.3 gives 1024

$$-(M-1) > v_j(p) = \int_0^1 r^{M-1} (\psi'_{j,p})^2 dr - \int_0^1 r^{M-3} f_p(r) \psi_{j,p}^2 dr$$
$$\geq -\int_0^1 r^{M-3} f_p(r) \psi_{j,p}^2 dr.$$

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To check (3.27) we begin by taking any $\varepsilon > 0$, applying Lemma 2.14 and splitting the integral 1026 1027 as

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$$\int_{0}^{1} r^{M-3} f_{p} \psi_{j,p}^{2} dr = \int_{0}^{K(\widetilde{\mathcal{M}}_{0,p})^{-1}} r^{M-3} f_{p} \psi_{j,p}^{2} dr + \sum_{i=1}^{m-1} \int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}}^{K(\widetilde{\mathcal{M}}_{i,p})^{-1}} r^{M-3} f_{p} \psi_{j,p}^{2} dr + \sum_{i=1}^{m} \int_{G_{i,p}(K)} r^{M-3} f_{p} \psi_{j,p}^{2} dr$$

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where K (and consequently $G_{i,p}(K)$) is chosen in such a way to satisfy (2.47). So using also 1031 (3.14) we obtain 1032

$$\sum_{i=1}^{m} \int_{G_{i,p}(K)} r^{M-3} f_p \psi_{j,p}^2 dr < \varepsilon \int_0^1 r^{M-3} \psi_{j,p}^2 dr = \varepsilon.$$

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On the other hand exploiting the uniform convergence stated in (3.18) we also have 1034

$$\int_{0}^{K(\widetilde{\mathcal{M}}_{0,p})^{-1}} r^{M-3} f_{p} \psi_{j,p}^{2} dr = p \int_{0}^{K(\widetilde{\mathcal{M}}_{0,p})^{-1}} r^{M-1} |v_{p}|^{p-1} \psi_{j,p}^{2} dr$$
$$= \int_{0}^{K} s^{M-1} W_{p}^{0} (\widetilde{\psi}_{j,p}^{0})^{2} ds \to \int_{0}^{K} s^{M-1} W (\widetilde{\psi}_{j}^{0})^{2} ds = 0$$

by (3.26), and similarly 1039

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$$\int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}}^{K(\widetilde{\mathcal{M}}_{i,p})^{-1}} r^{M-3} f_p \psi_{j,p}^2 \, dr = p \int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}}^{K(\widetilde{\mathcal{M}}_{i,p})^{-1}} r^{M-1} |v_p|^{p-1} \psi_{j,p}^2 \, dr$$
$$= \int_{K^{-1}}^{K} s^{M-1} W_p^i (\widetilde{\psi}_{j,p}^i)^2 \, ds \to \int_{K^{-1}}^{K} s^{M-1} W(\widetilde{\psi}_j^i)^2 \, ds = 0.$$

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Summing up we have proved that $\limsup_{p \to p_M} \int_0^1 r^{M-3} f_p \psi_{j,p}^2 dr < \varepsilon$ for every positive ε which 1043 clearly gives (3.27) since $f_p \ge 0$. 1044

3.2 The last negative eigenvalue 1045

As mentioned before, the last negative eigenvalue $v_m(p)$ has a different behavior from the 1046 first m-1 ones, which is enlightened by the different global bounds (3.9) and (3.10). In the 1047 case of Lane–Emden equation studied in [18] the relation (3.10) is sufficient to determine 1048 its contribution to the Morse index, therefore there is no need for further investigation. This 1049 is not the case anymore for the Hénon equation, where the exact computation of its limit is 1050 necessary to compute the asymptotic Morse index. 1051

To this aim a more detailed knowledge of the asymptotic behavior of the previous m-11052 eigenfunctions may help. 1053

Lemma 3.7 For every $\delta > 0$ there exist $\overline{K} > 1$ and $\overline{p} \in (1, p_M)$ such that 1054

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$$\int_{G_{i,p}(K)} r^{M-3} \psi_{j,p}^2 dr < \delta \quad \text{for } i = 1, \dots m, \, j = 1, \dots m-1,$$
(3.28)

for every $K > \overline{K}$ and $p \in (\overline{p}, p_M)$. 1056

Here $G_{i,p}(K) = \left(K(\widetilde{\mathcal{M}}_{i-1,p})^{-1}, (K\widetilde{\mathcal{M}}_{i,p})^{-1}\right)$ denotes the subset of (0, 1) introduced in Lemma 2.14. 1057 Lemma 2.14. 1058

Proof Let $\varepsilon \in (0, 1/2)$. By Lemma 2.14 we can choose $\bar{K}_1(\varepsilon)$ and $\bar{p}_1 = p_1(\varepsilon, \bar{K}_1)$ in such 1059 a way that for every $K > \overline{K_1}$ and $p \in (\overline{p_1}, p_M)$ we have 1060

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$$\int_{G_{i,p}(K)} r^{M-3} f_p \psi_{j,p}^2 dr < \varepsilon \int_{G_{i,p}(K)} r^{M-3} \psi_{j,p}^2 \leq \varepsilon$$
(3.29)

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.: 2019/9/5 Pages: 47 Layout: Small

for i = 1, ..., m and j = 1, ..., m - 1. Hence multiplying Eq. (3.4) for $\psi_{j,p}$ and integrating 1062 over $G_{i,p}(K)$ yields 1063

Next we write 1067

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$$\alpha = K(\mathcal{M}_{i-1,p})^{-1} \text{ for } i = 1, \dots m,$$

either $\beta = (K\widetilde{\mathcal{M}}_{i,p})^{-1} \text{ for } i = 1, \dots m-1, \text{ or } \beta = 1 \text{ if } i = m.$

so that $G_{i,p}(K) = (\alpha, \beta)$ and integrating by parts we have 1069

$$\int_{G_{i,p}(K)} (r^{M-1}\psi'_{j,p})'\psi_{j,p}dr = -\int_{G_{i,p}(K)} r^{M-1}(\psi'_{j,p})^2 dr + \beta^{M-1}\psi'_{j,p}(\beta)\psi_{j,p}(\beta) - \alpha^{M-1}\psi'_{i,p}(\alpha)\psi_{j,p}(\alpha).$$

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But by the definition of $\tilde{\psi}_{i,p}$ we have either 1073

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$$\alpha^{M-1}\psi'_{j,p}(\alpha)\psi_{j,p}(\alpha) = K^{M-1}\widetilde{\psi}_{j,p}^{i-1}(K)\,(\widetilde{\psi}_{j,p}^{i-1})'(K),$$

$$\beta^{M-1}\psi'_{j,p}(\beta)\psi_{j,p}(\beta) = K^{1-M}\widetilde{\psi}^{i}_{j,p}(K^{-1})(\widetilde{\psi}^{i}_{j,p})'(K^{-1}).$$

if i = 1, ..., m - 1, or 1077

$$\beta^{M-1}\psi'_{j,p}(\beta)\psi_{j,p}(\beta) = 1$$

if i = m. Therefore the convergence in Proposition 3.6 implies that when $p \rightarrow p_M$ either 1080

$$\beta^{M-1} \psi'_{j,p}(\beta) \psi_{j,p}(\beta) - \alpha^{M-1} \psi'_{j,p}(\alpha) \psi_{j,p}(\alpha) \rightarrow (A^{i}_{j})^{2} K^{1-M} \eta_{1}(K^{-1}) \eta'_{1}(K^{-1}) - (A^{i-1}_{j})^{2} K^{M-1} \eta_{1}(K) \eta'_{1}(K)$$

if i = 1, ..., m - 1, or 1084

$$\rightarrow -(A_j^{m-1})^2 K^{M-1} \eta_1(K) \, \eta_1'(K)$$

if i = m. Besides there exists $\bar{K} \ge \bar{K}_1$ so that for any $K > \bar{K}$ 1087

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$$-\varepsilon < K^{M-1}\eta_1(K)\eta'_1(K) < K^{1-M}\eta_1(K^{-1})\eta'_1(K^{-1}) < \varepsilon.$$
 (3.31)

This choice is possible because η_1 has only one critical point, which is a maximum, and 1089 $\eta_1(t), t^{M-1}\eta_1(t)\eta_1'(t) \to 0 \text{ as } t \to 0 \text{ and } t \to \infty.$ Then we can choose $p_2 = p_2(\varepsilon, \bar{K})$ in 1090 such a way that 1091

$$\int_{G_{i,p}(K)} (r^{M-1}\psi'_{j,p})'\psi_{j,p}dr < -\int_{G_{i,p}(K)} r^{M-1}(\psi'_{j,p})^2 dr + A\varepsilon \le A\varepsilon$$

for $p \in (p_2, p_M)$ for any K > K. Here the constant A only depends by the coefficients A_i^i . 1094 Inserting this bound into (3.30) gives 1095

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$$-\nu_{j}(p)\int_{G_{i,p}(K)}r^{M-3}\psi_{j,p}^{2}dr < (1+A)\varepsilon$$

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in the same range of the parameter p. Moreover 3.6 yields that also $-v_i(p) > (M-1)(1-\varepsilon)$, 1008 possibly increasing p_2 . Hence recalling that $\varepsilon < 1/2$ we get 1099

$$\int_{G_{i,p}(K)} r^{M-3} \psi_{j,p}^2 dr < \frac{1+A}{M-1} \frac{\varepsilon}{1-\varepsilon} \le C\varepsilon$$

where C only depends by A and M, and this concludes the proof. 1101

Lemma 3.8 The constants A_i^i in Proposition 3.6 satisfy 1102

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$$\sum_{i=0}^{m-1} A_j^i A_k^i \int_0^{+\infty} r^{M-3} \eta_1^2 dr = \delta_{jk}$$
(3.32)

for every j, k = 1, ..., m - 1. 1104

Proof Let 1105

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$$H(p) := \int_0^1 r^{M-3} \psi_{j,p} \psi_{k,p} dr - \sum_{i=0}^{m-1} A^i_j A^i_k \int_0^{+\infty} r^{M-3} \eta_1^2 dr.$$

By (3.12) we have 1107

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$$\delta_{jk} - \sum_{i=0}^{m-1} A_j^i A_k^i \int_0^{+\infty} r^{M-3} \eta_1^2 dr = H(p)$$

for every $p \in (1, p_M)$, and the claim can be proved by showing that $H(p_n) \to 0$ for the 1109 sequence p_n which realizes 1110

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 $\widetilde{\psi}^i_{j,p_n} \to A^i_j \eta_1$

for $i = 0, \dots, m-1$ and $j = 1, \dots, m-1$, according to Proposition 3.6. More precisely we 1112 will show that for any $\varepsilon > 0$ we can choose \bar{n} in such a way that $|H(p_n)| < \varepsilon$ as $n > \bar{n}$. Not to 1113 make notation even heavier, in the following we shall write p meaning p_n , and $p \in (\bar{p}, p_M)$ 1114 meaning $n > \bar{n}$. 1115

Let K > 1 be a parameter to be chosen later on according to ε ; we split the interval (0, 1)1116 in the same way used in Lemma 2.14 and write 1117

H(p) =
$$\sum_{i=1}^{m} \int_{G_{i,p}(K)} r^{M-3} \psi_{j,p} \psi_{k,p} dr + \int_{0}^{K(\widetilde{\mathcal{M}}_{0,p})^{-1}} r^{M-3} \psi_{j,p} \psi_{k,p} dr$$

+ $\sum_{i=1}^{m-1} \int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}} r^{M-3} \psi_{j,p} \psi_{k,p} dr - \sum_{i=0}^{m-1} A_{j}^{i} A_{k}^{i} \int_{0}^{+\infty} r^{M-3} \eta_{1}^{2} dr.$

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$$+\sum_{i=1}^{N}\int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}}$$

Now 1121

$$2 \qquad \left| \int_{G_{i,p}(K)} r^{M-3} \psi_{j,p} \psi_{k,p} dr \right| \le \left(\int_{G_{i,p}(K)} r^{M-3} \psi_{j,p}^2 dr \right)^{\frac{1}{2}} \left(\int_{G_{i,p}(K)} r^{M-3} \psi_{k,p}^2 dr \right)^{\frac{1}{2}},$$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

Page 35 of 47 _####_

so Lemma 3.7 yields that we can choose $\bar{K}_0 = \bar{K}_0(\varepsilon)$ and $\bar{p}_0 = \bar{p}_0(\varepsilon, K_0)$ in such a way that

 $\left| \int_{G_{i,p}(K)} r^{M-3} \psi_{j,p} \psi_{k,p} dr \right| < \varepsilon/3m$ (3.33)

1126 for $K \ge \bar{K}_0$ and $p \in (\bar{p}_0, p_M)$.

Besides rescaling and using the convergence in Proposition 3.6, it is easy to see that for every K

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$$\int_{0}^{K(\widetilde{\mathcal{M}}_{0,p})^{-1}} r^{M-3}\psi_{j,p}\psi_{k,p}dr = \int_{0}^{K} r^{M-3}\widetilde{\psi}_{j,p}^{0}\widetilde{\psi}_{k,p}^{0}dr \to A_{j}^{0}A_{k}^{0}\int_{0}^{K} r^{M-3}\eta_{1}^{2}dr$$

1130 as $p \to p_M$, as well as

$$\int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}}^{K(\widetilde{\mathcal{M}}_{i,p})^{-1}} r^{M-3}\psi_{j,p}\psi_{k,p}dr = \int_{K^{-1}}^{K} r^{M-3}\widetilde{\psi}_{j,p}^{i}\widetilde{\psi}_{k,p}^{i}dr \to A_{j}^{i}A_{k}^{i}\int_{K^{-1}}^{K} r^{M-3}\eta_{1}^{2}dr$$

for i = 1, ..., m - 1. Since $r^{M-3}\eta_1^2 \in L^1(0, \infty)$, there exists $\bar{K}_1 = \bar{K}_1(\varepsilon) > 1$ such that

$$|A_{j}^{0}A_{k}^{0}| \int_{K}^{\infty} r^{M-3}\eta_{1}^{2}dr + \sum_{i=1}^{m-1} |A_{j}^{i}A_{k}^{i}| \left(\int_{0}^{K^{-1}} r^{M-3}\eta_{1}^{2}dr + \int_{K}^{\infty} r^{M-3}\eta_{1}^{2}dr\right) < \varepsilon/3$$

as $K > \bar{K}_1$ and consequently for any $K > \bar{K}_1$ we can choose $p_1 = p_1(\varepsilon, K)$ in such a way that

$$\begin{vmatrix} K(\widetilde{\mathcal{M}}_{0,p})^{-1} & m^{-1} \\ \int_{0}^{m-1} r^{M-3} \psi_{j,p} \psi_{k,p} dr + \sum_{i=1}^{m-1} \int_{(K\widetilde{\mathcal{M}}_{i,p})^{-1}}^{K(\widetilde{\mathcal{M}}_{i,p})^{-1}} r^{M-3} \psi_{j,p} \psi_{k,p} dr \\ - \sum_{i=0}^{m-1} A_{j}^{i} A_{k}^{i} \int_{0}^{+\infty} r^{M-3} \eta_{1}^{2} dr \end{vmatrix} < 2\varepsilon/3$$
(3.34)

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 $\sum_{i=0}^{2} \int \left(x \int \right) \left(x \int \right) = \left(x \int \right)$

for every $p \in (p_1, p_M)$. Putting together (3.33) and (3.34) gives the claim.

1138 **Corollary 3.9** There exists an index $k \in \{0, 1, \dots, m-1\}$ such that

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$$\sum_{j=1}^{m-1} (A_j^k)^2 < \left(\int_0^\infty t^{M-3} \eta_1^2 dt\right)^{-1}$$

Proof Let $C = \left(\int_0^\infty t^{M-3} \eta_1^2 dt\right)^{-1}$. Using (3.32) with j = k we immediately have

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$$\sum_{i=0}^{m-1} (A_j^i)^2 = C$$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

1142 for every $j = 1, \dots, m - 1$. Therefore

Author Proof

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$$\sum_{i=0}^{m-1} \left(\sum_{j=1}^{m-1} (A_j^i)^2 \right) = \sum_{j=1}^{m-1} \left(\sum_{i=0}^{m-1} (A_j^i)^2 \right) = (m-1)C.$$

Since all the *m* terms $\sum_{j=1}^{m-1} (A_j^i)^2$ are nonnegative, at least one among them should satisfy

$$\sum_{j=1}^{m-1} (A_j^i)^2 \le \frac{m-1}{m}C < C$$

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Such index k will play a role in the proof of next proposition, which is the main result in the present subsection.

1149 **Proposition 3.10** *We have*

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$$\lim_{p \to p_M} \nu_m(p) = -(M-1).$$

Moreover for any sequence p_n in $(1, p_M)$ with $p_n \to p_M$ there exist a subsequence (that we still denote by p_n), and m numbers $A_m^0, \ldots A_m^{m-1} \in \mathbb{R}$ such that

$$\widetilde{\psi}^i_{m,p_n} o A^i_m \eta_1$$

weakly in $\mathcal{D}_M(0,\infty)$ and strongly in $C^1_{\text{loc}}(0,\infty)$.

Proof By virtue of Lemma 3.4 and Remark 3.5 it is enough to show that

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Moreover, thanks to (3.10), it suffices to check that

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$$\limsup_{p \to p_M} \nu_m(p) \le -(M-1).$$

 $\lim_{p\to p_M} \nu_m(p) = -(M-1).$

We therefore choose a sequence $p_n \to p_M$ such that $\nu_m(p_n) \to \limsup_{p \to p_M} \nu_m(p)$. Possibly

passing to a subsequence, we may assume w.l.g. that $\tilde{\psi}_{j,p_n}^i \to A_j^i \eta_1$ for $i = 0, \dots m - 1$ and $j = 1, \dots m - 1$, in force of Proposition 3.6. Not to make notation even heavier, in the following we shall write p, meaning p_n .

Now the claim follows by producing, for every $\varepsilon > 0$, a family of nontrivial test functions $\psi_p \in H^1_{0,M}, \psi_p \perp_M \{\psi_{1,p}, \dots, \psi_{m-1,p}\}$, such that

$$\lim_{p \to p_M} \sup_{\mathcal{R}_p(\psi)} \mathcal{R}_p(\psi) \leq -(M-1) + \varepsilon,$$

$$\mathcal{R}_p(\psi) := \frac{\int_0^1 r^{M-1} (\psi')^2 - r^{M-3} f_p(r) \psi^2 dr}{\int_0^1 r^{M-3} \psi^2 dr},$$
(3.35)

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and recalling the variational characterization (3.6).

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Let us consider the index k in Corollary 3.9 and define

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$$\psi_p(r) := (\eta_1 \Phi) \left(r \widetilde{\mathcal{M}}_{k,p} \right) + \sum_{j=1}^{m-1} a_{j,p} \psi_{j,p}(r),$$

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where $\Phi \in C_0^{\infty}(0, \infty)$ is a cut-off function with 1169

$$0 \le \Phi(r) \le 1$$
, for every $r \in [0, \infty)$, (3.36)

$$\Phi(r) = \begin{cases} 0 & \text{if } r \in [0, (2R)^{-1}] \text{ or } [2R, \infty), \\ 1 & \text{if } r \in [R^{-1}, R] \end{cases}$$
(3.37)

$$(r) = \begin{cases} 1 & \text{if } r \in [R^{-1}, R], \end{cases}$$
(3.3)

$$|\Phi'(r)| \le \begin{cases} 2R & \text{if } r \in [(2R)^{-1}, R^{-1}], \\ 2R^{-1} & \text{if } r \in [R, 2R] \end{cases}$$
(3.38)

and η_1 as defined in (3.20). Here R is a parameter to be suitably chosen, depending on ε . 1174 Since we will send $p \rightarrow p_M$, thanks to (2.33) and (2.35) we may take w.l.g. that 1175

$$t_{k,p}\widetilde{\mathcal{M}}_{k,p} < (2R)^{-1} < 2R < t_{k+1,p}\widetilde{\mathcal{M}}_{k,p} \le \widetilde{\mathcal{M}}_{k,p}.$$
(3.39)

The coefficients $a_{j,p}$, instead, are chosen in such a way to ensure that $\psi_p \perp_M$ 1177 $\{\psi_{1,p},\ldots,\psi_{m-1,p}\}$ for every p, namely 1178

$$a_{j,p} = -\int_0^1 r^{M-3} \psi_{j,p}(r)(\eta_1 \Phi) \big(r \widetilde{\mathcal{M}}_{k,p} \big) dr.$$

By (3.37) and (3.39) we have 1181

$$a_{j,p} = -\int_{l_{k,p}}^{l_{k+1,p}} r^{M-3} \psi_{j,p}(r)(\eta_1 \Phi) \big(r \widetilde{\mathcal{M}}_{k,p} \big) dr,$$

so performing the change of variables $t = r \widetilde{\mathcal{M}}_{k,p}$ and recalling the definition of $\widetilde{\psi}_{i,p}^{k}$ in 1184 (3.13) one gets 1185

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$$a_{j,p} = -(\widetilde{\mathcal{M}}_{k,p})^{-\frac{M-2}{2}} \int_{t_{k,p}\widetilde{\mathcal{M}}_{k,p}}^{t_{k+1,p}\widetilde{\mathcal{M}}_{k,p}} t^{M-3}\widetilde{\psi}_{j,p}^{k} \eta_1 \Phi dt = (\widetilde{\mathcal{M}}_{k,p})^{-\frac{M-2}{2}} \widetilde{a}_{j,p}$$
1187

1188 for

1189 1190

$$\widetilde{a}_{j,p} = -\int_0^{+\infty} t^{M-3} \widetilde{\psi}_{j,p}^k \eta_1 \Phi dt$$

Obtaining (3.35) will request many computations, that we split in several claims. 1191 Claim 1:

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$$\mathcal{D}(p) := \int_0^1 r^{M-3} \psi_p^2 dr$$

= $(\widetilde{\mathcal{M}}_{k,p})^{2-M} \left[\int_0^\infty t^{M-3} (\eta_1 \Phi)^2 (t) dt - \sum_{j=1}^{m-1} (\widetilde{a}_{j,p})^2 \right].$ (3.40)

It suffices to compute 1194

$$\mathcal{D}(p) = \int_{0}^{1} r^{M-3} (\eta_{1} \Phi)^{2} \left(r \widetilde{\mathcal{M}}_{k,p} \right) dr + \sum_{j,k=1}^{m-1} a_{j,p} a_{k,p} \int_{0}^{1} r^{M-3} \psi_{j,p} \psi_{k,p} dr$$

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 $+2\sum_{i=1}a_{j,p}\int_0^1r^{M-3}\psi_{j,p}\left(\eta_1\Phi\right)\left(r\widetilde{\mathcal{M}}_{k,p}\right)dr,$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

where performing the change of variables $t = r \widetilde{\mathcal{M}}_{k,p}$ in the first integral and taking advantage 1198 from (3.37) and (3.39) we have 1199

$$\int_{0}^{1} r^{M-3} (\eta_{1} \Phi)^{2} (r \widetilde{\mathcal{M}}_{k,p}) dr = (\widetilde{\mathcal{M}}_{k,p})^{2-M} \int_{0}^{\widetilde{\mathcal{M}}_{k,p}} t^{M-3} (\eta_{1} \Phi)^{2} (t) dt$$
$$= (\widetilde{\mathcal{M}}_{k,p})^{2-M} \int_{0}^{\infty} t^{M-3} (\eta_{1} \Phi)^{2} (t) dt.$$

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Author Proof

Next using (3.12) and the definition of $a_{j,p}$, $\tilde{a}_{j,p}$ in the second and third integrals gives 1203 (3.40). Further Claim 2: 1204

$$\mathcal{N}_{1}(p) := \int_{0}^{1} r^{M-3} f_{p} \psi_{p}^{2} dr = (\widetilde{\mathcal{M}}_{k,p})^{2-M} \int_{0}^{\infty} t^{M-3} \widetilde{f}_{k,p}(t) (\eta_{1} \Phi)^{2} (t) dt + \sum_{j \ k=1}^{m-1} a_{j,p} a_{k,p} \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p} \psi_{k,p} dr$$

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$$+2\sum_{j=1}^{m-1}a_{j,p}\int_{0}^{1}r^{M-3}f_{p}(r)\psi_{j,p}(r)\left(\eta_{1}\Phi\right)\left(r\widetilde{\mathcal{M}}_{k,p}\right)dr$$
(3.41)

where $\widetilde{f}_{k,p}$ is as defined in (2.43). Indeed it suffices to write explicitly 1209

$$\mathcal{N}_{1}(p) = \int_{0}^{1} r^{M-3} f_{p}(r) \left(\eta_{1} \Phi\right)^{2} \left(r \widetilde{\mathcal{M}}_{k,p}\right) dr + \sum_{j,k=1}^{m-1} a_{j,p} a_{k,p} \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p} \psi_{k,p} dr$$

$$+2\sum_{j=1}^{m-1}a_{j,p}\int_{0}^{1}r^{M-3}f_{p}(r)\psi_{j,p}(r)(\eta_{1}\Phi)(r\widetilde{\mathcal{M}}_{k,p})dr,$$

perform the change of variables $t = r \widetilde{\mathcal{M}}_{k,p}$ and taking again advantage from (3.37) and 1213 (3.39) in the first integral. 1214

Besides, Claim 3: 1215

$$\mathcal{N}_{2}(p) := \int_{0}^{1} r^{M-1} (\psi')^{2} dr = (\widetilde{\mathcal{M}}_{k,p})^{2-M} \left[-(M-1) \int_{0}^{\infty} t^{M-3} (\eta_{1} \Phi)^{2} dt + \int_{0}^{\infty} t^{M-1} W(\eta_{1} \Phi)^{2} dt + \int_{0}^{\infty} t^{M-1} (\eta_{1} \Phi')^{2} dt - \sum_{i=1}^{m-1} v_{j}(p) (\widetilde{a}_{j,p})^{2} \right]$$

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$$+ \sum_{j,k=1}^{m-1} a_{j,p} a_{k,p} \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p} \psi_{k,p} dr + 2 \sum_{j=1}^{m-1} a_{j,p} \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p} (\eta_{1} \Phi) (r \widetilde{\mathcal{M}}_{k,p}) dr.$$
(3.42)

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

1221 By definition

$$\mathcal{N}_{2}(p) = \int_{0}^{1} r^{M-1} \left(\left((\eta_{1} \Phi) \left(r \widetilde{\mathcal{M}}_{k,p} \right) \right)' \right)^{2} dr + \sum_{j,k=1}^{m-1} a_{j,p} a_{k,p} \int_{0}^{1} r^{M-1} \psi'_{j,p} \psi'_{k,p} dr + 2 \sum_{j=1}^{m-1} a_{j,p} \int_{0}^{1} r^{M-1} \psi'_{j,p} \left((\eta_{1} \Phi) \left(r \widetilde{\mathcal{M}}_{k,p} \right) \right)' dr$$

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As for the first term, we have

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$$\int_{0}^{1} r^{M-1} \left(\left((\eta_{1} \Phi) (r \widetilde{\mathcal{M}}_{k,p}) \right)' \right)^{2} dr = (\widetilde{\mathcal{M}}_{k,p})^{2} \int_{0}^{1} r^{M-1} \left((\eta_{1} \Phi)' (r \widetilde{\mathcal{M}}_{k,p}) \right)^{2} dr$$
$$= (\widetilde{\mathcal{M}}_{k,p})^{2-M} \int_{0}^{\infty} t^{M-1} \left((\eta_{1} \Phi)' \right)^{2} dt$$

after performing the change of variables $t = r \widetilde{\mathcal{M}}_{k,p}$ and recalling (3.37), (3.39). Next we decompose $((\eta_1 \Phi)')^2 = \eta'_1 (\eta_1 \Phi^2)' + (\eta_1 \Phi')^2$, so that

$$= (\widetilde{\mathcal{M}}_{k,p})^{2-M} \left(\int_0^\infty t^{M-1} \eta_1' \left(\eta_1 \Phi^2 \right)' dt + \int_0^\infty t^{M-1} (\eta_1 \Phi')^2 dt \right)$$

and remembering that η_1 is the first eigenfunction for (3.19) and solves (3.21) with $\beta_1 = -(M-1)$, we have

$$= (\widetilde{\mathcal{M}}_{k,p})^{2-M} \left(-(M-1) \int_0^\infty t^{M-3} (\eta_1 \Phi)^2 dt + \int_0^\infty t^{M-1} W(\eta_1 \Phi)^2 dt + \int_0^\infty t^{M-1} (\eta_1 \Phi')^2 dt \right).$$

Next (3.7) yields

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 $\int_{0}^{1} r^{M-1} \psi'_{j,p} \psi'_{k,p} dr = \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p} \psi_{k,p} dr + v_{j}(p) \delta_{jk}$

thanks to (3.12). Concerning the last term, Eq. (3.7) again gives

$$\int_{0}^{1} r^{M-1} \psi'_{j,p} \left((\eta_{1} \Phi) (r \widetilde{\mathcal{M}}_{k,p}) \right)' dr = \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p}(r) (\eta_{1} \Phi) (r \widetilde{\mathcal{M}}_{k,p}) dr$$

$$+ v_{j}(p) \int_{0}^{1} r^{M-3} \psi_{j,p}(r) (\eta_{1} \Phi) (r \widetilde{\mathcal{M}}_{k,p}) dr$$

$$= \int_{0}^{1} r^{M-3} f_{p} \psi_{j,p}(r) (\eta_{1} \Phi) (r \widetilde{\mathcal{M}}_{k,p}) dr - v_{j}(p) a_{j,p}(r) dr$$

¹²⁴⁶ So the claim follows after summing up the three terms.

Adding (3.40), (3.41) and (3.42) gives

$$\mathcal{R}_p(\psi_p) = \frac{\mathcal{N}_2(p) - \mathcal{N}_1(p)}{\mathcal{D}(p)} = -(M-1) + \frac{\mathcal{A}_p(\Phi)}{\mathcal{B}_p(\Phi)}$$

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Author Proof

where 1250

$$\mathcal{A}_{p}(\Phi) = \int_{0}^{\infty} t^{M-3} (t^{2}W - \tilde{f}_{k,p}) (\eta_{1}\Phi)^{2} dt + \int_{0}^{\infty} t^{M-1} (\eta\Phi')^{2} dt - \sum_{j=1}^{m-1} (\nu_{j}(p) + M - 1) (\tilde{a}_{j,p})^{2}$$

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But when $p \to p_M$, then $\widetilde{f}_{k,p} \to F = t^2 W$ uniformly on $[R^{-1}, R]$ by Lemma 2.12, so that

$$\int_0^\infty t^{M-3} \left(t^2 W - \widetilde{f}_{k,p} \right) \left(\eta_1 \Phi \right)^2 dt \to 0.$$

 $\mathcal{B}_{p}(\Phi) = \int_{0}^{\infty} t^{M-3} (\eta_{1}\Phi)^{2} dt - \sum_{i=1}^{m-1} (\widetilde{a}_{j,p})^{2}$

Besides Proposition 3.6 assures that $v_i(p) + M - 1 \rightarrow 0$ and that 1257

$$\widetilde{a}_{j,p} = -\int_0^{+\infty} t^{M-3} \widetilde{\psi}_{j,p}^k \eta_1 \Phi dt \to -A_j^k \int_0^{+\infty} t^{M-3} \eta_1^2 \Phi dt$$

as $p \rightarrow p_M$. Therefore 1260

$$\lim_{p \to p_M} \mathcal{R}_p(\psi_p) = -(M-1)$$

$$+ \frac{\int_0^{\infty} t^{M-1} (\eta_1 \Phi)^2 dt}{\int_0^{\infty} t^{M-3} (\eta_1 \Phi)^2 dt - \left(\int_0^{+\infty} t^{M-3} \eta_1^2 \Phi dt\right)^2 \sum_{j=1}^{m-1} (A_j^k)^2}$$

We conclude the proof by showing that for every $\varepsilon > 0$ it is possible to choose R and the 1264 cut-off function Φ satisfying (3.36)–(3.38) in such a way that 1265

$$\frac{\int_0^\infty t^{M-1} (\eta_1 \Phi')^2 dt}{\int_0^\infty t^{M-3} (\eta_1 \Phi)^2 dt - \left(\int_0^{+\infty} t^{M-3} \eta_1^2 \Phi dt\right)^2 \sum_{j=1}^{m-1} (A_j^k)^2} < \varepsilon.$$

To begin with 1267

$$\int_0^\infty t^{\infty}$$

$$\int_{0}^{\infty} t^{M-1} (\eta_{1} \Phi')^{2} dt = \int_{\frac{1}{2R}}^{\frac{1}{R}} t^{M-1} (\eta_{1} \Phi')^{2} dt + \int_{R}^{2R} t^{M-1} (\eta \Phi')^{2} dt$$

$$\leq CR^{2} \int_{\frac{1}{2R}}^{\frac{1}{R}} t^{M-1} \eta_{1}^{2} dt + \frac{C}{R^{2}} \int_{R}^{2R} t^{M-1} \eta_{1}^{2} dt$$

and since η_1 has a unique maximum point in $\bar{t} \in (0, +\infty)$, if $R > \max\{\bar{t}, 1/\bar{t}\}$ we have 1271

$$= \frac{CR^2 \left(\eta_1\left(\frac{1}{R}\right)\right)^2 \int_{\frac{1}{2R}}^{\frac{1}{R}} t^{M-1} dt + \frac{C}{R^2} \left(\eta_1(2R)\right)^2 \int_{R}^{2R} t^{M-1} dt }{\frac{C^2(1-2^{-M})}{MR^M \left(1+\frac{1}{1+\frac{1}{1+\frac{1}{2R}}}\right)^M} + \frac{C^2(2^M-1)R^M}{M \left(1+\frac{R^2}{1+\frac{R^2}}\right)^M} = o(1)$$

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$$= \frac{1}{MR^{M} \left(1 + \frac{1}{M(M-2)R^{2}}\right)^{M}} + \frac{1}{M \left(1 + \frac{R^{2}}{M(M-2)}\right)^{M}}$$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

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1275 as $R \to \infty$. Next it is clear that

$$\int_0^\infty t^{M-3} (\eta_1 \Phi)^2 dt \to \int_0^\infty t^{M-3} \eta_1^2 dt > 0$$

1277 as $R \to \infty$, because

$$0 \leq \int_0^\infty t^{M-3} \eta_1^2 dt - \int_0^\infty t^{M-3} (\eta_1 \Phi)^2 dt$$

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$$= \int_{(3.37)}^{\frac{1}{R}} t^{M-3} \eta_1^2 (1 - \Phi^2) dt + \int_R^{\infty} t^{M-3} \eta_1^2 (1 - \Phi^2) dt$$

$$\leq \int_{(3.36)}^{\frac{1}{R}} t^{M-3} \eta_1^2 dt + \int_R^{\infty} t^{M-3} \eta_1^2 dt = o(1)$$

since $\int_0^\infty t^{M-3} \eta_1^2 dt < \infty$. Similarly

$$\int_0^\infty t^{M-3}\eta_1^2\Phi dt \to \int_0^\infty t^{M-3}\eta_1^2 dt > 0.$$

1284 Eventually

$$\int_{0}^{\infty} t^{M-3} (\eta_{1} \Phi)^{2} dt - \left(\int_{0}^{+\infty} t^{M-3} \eta_{1}^{2} \Phi dt\right)^{2} \sum_{j=1}^{m-1} (A_{j}^{k})^{2}$$

$$\rightarrow \int_{0}^{\infty} t^{M-3} \eta_{1}^{2} dt - \left(\int_{0}^{\infty} t^{M-3} \eta_{1}^{2} dt\right)^{2} \sum_{j=1}^{m-1} (A_{j}^{k})^{2} \neq 0$$

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¹²⁸⁶ by Corollary 3.9, which ends the proof.

¹²⁸⁷ We are now in position to prove Theorem 1.1.

Proof Propositions 3.6 and 3.10 prove that each generalized radial singular negative eigen-1288 value $\widehat{v}_i(p) \to -(M-1)$ as $p \to p_M$ for $i = 1, \dots, m$. Inserting these asymptotic values 1289 into (3.11) gives that $J_i(p) \to 1 + \frac{\alpha}{2}$ as $p \to p_{\alpha} = p_M$ for j = 1, ..., m. In particular from 1290 (3.9) and (3.10) we have $J_i(p) \nearrow 1 + \frac{\alpha}{2}$ for j = 1, ..., m-1 while $J_m(p) \searrow 1 + \frac{\alpha}{2}$. Then, 1291 when α is not an even integer all the eigenvalues $\hat{v}_i(p)$ gives the same contribution to the 1292 Morse index giving (1.4). When α is an even integer instead in the sum in (3.11) we have to 1293 add the contribution of all the *m* eigenvalues for $j \leq \frac{\alpha}{2}$ and the contribution of only m-11294 eigenvalues for $j = 1 + \frac{\alpha}{2}$, which gives (1.5). 1295

1296 4 Nondegeneracy and small perturbations

In this section we address the nondegeneracy of radial solutions to (1.1) when *p* approaches p_{α} and we prove Theorem 1.3 and its consequence Theorem 1.4. We recall that a solution *u* to (1.1) is said nondegenerate if the linearized operator at *u*, L_u , does not admit zero as an eigenvalue in $H_0^1(B)$, and hence if the linearized equation at *u*, namely

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$$\begin{cases} -\Delta \psi = p|x|^{\alpha}|u|^{p-1}\psi & \text{in } B, \\ \psi = 0 & \text{on } \partial B, \end{cases}$$
(4.1)

i=1

does not admit any nontrivial solution in $H_0^1(B)$. Degeneracy can be computed by analyzing the singular Sturm-Liouville eigenvalue problem related to the transformed function v_p

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introduced in (2.2) as in the previous section. Indeed degeneracy of radial solutions to (1.1)1304 has been characterized in [3] using the singular negative radial eigenvalues $\hat{v}_k(p)$, defined in 1305 (3.6), for k = 1, ..., m. Putting together Proposition 1.5 of [3] and Theorem 1.3 of [5] we 1306 obtain 1307

Proposition 4.1 Let $\alpha \geq 0$ and $p \in (1, p_{\alpha})$. A radial solution u_p to (1.1) with m nodal zones 1308 is radially nondegenerate and it is degenerate if and only 1309

$$\widehat{\nu}_k(p) = -\left(\frac{2}{2+\alpha}\right)^2 j(N-2+j)$$

for some $k = 1, \ldots, m$ and for some $j \ge 1$. 1311

Therefore the asymptotic nondegeneracy of u_p as $p \rightarrow p_{\alpha}$ can be deduced, via the 1312 transformation (2.2), by the asymptotic behavior of the radial singular eigenvalues $\hat{v}_k(p)$ as 1313 $p \rightarrow p_M$. Indeed by the analysis performed in Sect. 3 we have: 1314

Proof of Theorem 1.3 Let us denote by g(s) the decreasing function 1315

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$$g(s) := -s(N - 2 + s), \quad s \ge 0.$$

By Proposition 4.1 u_p is degenerate if and only if there is some k = 1, ..., m such that 1317

$$\left(\frac{2+\alpha}{2}\right)^2 \widehat{\nu}_k(p) = g(j) \quad \text{for some positive integer } j. \tag{4.2}$$

Recalling that $-(M-1) = -\frac{2}{2+\alpha} \left(N-2+\frac{2+\alpha}{2}\right)$ according to (2.4), Propositions 3.6 and 3.10 imply that 1319 and 3.10 imply that 1320

$$\left(\frac{2+\alpha}{2}\right)^2 \widehat{\nu}_k(p) \to g\left(\frac{2+\alpha}{2}\right) \quad \text{for every } k = 1, \dots m \tag{4.3}$$

as $p \to p_M$. Therefore if α is not a nonnegative even integer, it is easily seen that 1322

$$(\frac{2+\alpha}{2})^2 \,\widehat{\nu}_k(p) \in \left(g\left(2+\left\lfloor\frac{\alpha}{2}\right\rfloor\right), g\left(1+\left\lfloor\frac{\alpha}{2}\right\rfloor\right)\right) \quad \text{for every } k=1,\dots,m$$

in a left neighborhood of p_M , which ensures that (4.2) can not hold since g is strictly 1324 decreasing. 1325

Otherwise when $\alpha = 2(j-1)$, then (4.3) says that $\left(\frac{2+\alpha}{2}\right)^2 \widehat{\nu}_k(p) \to g(j)$, but (3.9) and 1326 (3.10) imply that 1327

$$\begin{pmatrix} \left(\frac{2+\alpha}{2}\right)^2 \widehat{\nu}_k(p) < g(j) \quad \text{for } k = 1, \dots m-1,$$

$$\begin{pmatrix} \left(\frac{2+\alpha}{2}\right)^2 \widehat{\nu}_m(p) > g(j),$$
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for every $p \in (1, p_M)$. Therefore 1331

$$\begin{pmatrix} \frac{2+\alpha}{2} \end{pmatrix}^2 \widehat{\nu}_k(p) \in (g(j+1), g(j)) \quad \text{for } k = 1, \dots m-1,$$

$$\begin{pmatrix} \frac{2+\alpha}{2} \end{pmatrix}^2 \widehat{\nu}_m(p) \in (g(j), g(j-1))$$

in a left neighborhood of p_M , and the conclusion follows by the monotonicity of g, again. \Box 1335

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Asymptotic profile and Morse index of nodal radial solutions to...

As said before the nondegeneracy of u_p has important applications. Among them, we mention a procedure introduced by Davila and Dupaigne in [17] which allows one to deduce existence results in domains which are perturbations of the ball. We quote also [15] and [6] for applications to the Hénon problem and to nodal solutions annular domains, respectively. Let $\sigma : \bar{B} \to \mathbb{R}^N$ be a smooth function and

$$\Omega_t := \{ x + t\sigma(x) : x \in B \}.$$

1342 We want to find solutions to

$$\begin{cases} -\Delta u = |x|^{\alpha} |u|^{p-1} u & \text{in } \Omega_t, \\ u = 0 & \text{on } \partial \Omega_t, \end{cases}$$
(4.4)

For small values of *t*, the set Ω_t is diffeomorphic to *B* and hence there exists $\tilde{\sigma} : \bar{\Omega}_t \to \mathbb{R}^N$ such that $x = y + t\tilde{\sigma}(y)$ for every $x \in B$ and every $y \in \Omega_t$. It was noticed in [15] that if u(y) is a classical solution to (4.4) then w(x) = u(y) is a classical solution to

$$\begin{cases} -\Delta w - L_t(w) = |x + t\sigma(x)|^{\alpha} |w|^{p-1} w & \text{in } B, \\ w = 0 & \text{on } \partial B, \end{cases}$$
(4.5)

1348 where L_t is the linear operator

$$L_t(w) := t \sum_{i,k} \partial_{y_i y_i}^2 \tilde{\sigma}_k \partial_{x_k} w + 2t \sum_{i,k} \partial_{y_i} \tilde{\sigma}_k \partial_{x_i x_k}^2 w + t^2 \sum_{i,j,k} \partial_{y_j} \tilde{\sigma}_i \partial_{y_j} \tilde{\sigma}_k \partial_{x_i x_k}^2 w$$

and $\tilde{\sigma}_k$ denotes the *k*-th component of $\tilde{\sigma}$. Observe that u_p solves (4.5) for t = 0.

By the nondegeneracy of u_p stated in Theorem 1.3 it is not hard to deduce the existence of nodal solutions in domains of type Ω_t , i.e. to prove our last result.

Proof of Theorem 1.4 When $\alpha = 0$ or $\alpha > 1$ the map

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$$F: \mathbb{R} \times C_0^{2,\gamma}(\bar{B}) \to C_0^{0,\gamma}(\bar{B}) \quad F(t,w) = -\Delta w - L_t w - |x + t\sigma|^{\alpha} |w|^{p-1} w$$

where $C_0^{2,\gamma}(\bar{B}) := \{ w \in C^{2,\gamma}(\bar{B}) : w_{|\partial B|} = 0 \}$, is of class C^1 for γ small enough, and 1355 clearly $F(0, u_p) = 0$, where u_p is the radial solution to (1.1). Moreover $D_w F(0, u_p)$ (the 1356 Fréchet derivative of F with respect to $w \in C_0^{2,\gamma}(\overline{B})$ computed at $(0, u_p)$ is nothing else 1357 than the linearized operator L_{u_p} , which is invertible for $p > \bar{p}$ appearing in the statement of 1358 Theorem 1.3, because its kernel is made up by the solutions of the linearized problem (4.1). 1359 So the Implicit Function Theorem applies giving a continuum of functions $w_t \in C_0^{2,\gamma}(\bar{B})$ 1360 such that $F(t, w_t) = 0$. In particular $u_t(y) := w_t(x)$ is a solution of (4.5), it has exactly m 1361 nodal zones and its nodal curves does not intersect the boundary, at least for small t, thanks 1362 to the continuity of the maps $t \mapsto w_t \in C_0^{2,\gamma}(\bar{B})$ and $x \to x + t\sigma(x)$. 1363

1364 5 Appendix

In the paper [21] Gidas studied with a phase plane analysis the problem

 $\begin{cases} -u'' - \frac{N-1}{r}u' = u^{\frac{N+2}{N-2}} & \text{in } (0, \infty) \\ u > 0 \end{cases}$

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Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small

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and proved that, for N > 2, the solutions can have the following shapes: 1367

a)
$$u(r) = \left(\frac{\lambda\sqrt{N(N-2)}}{\lambda^2 + r^2}\right)^{\frac{N-2}{2}}$$

where λ is a positive parameter, or 1370

b)
$$u(r) = \left(\frac{N-2}{2}\right)^{\frac{N-2}{2}} r^{-\frac{N-2}{2}},$$

c) $c_1 r^{-\frac{N-2}{2}} \le u(r) \le c_2 r^{-\frac{N-2}{2}}.$

When N is an integer it has later been proved that only case a) and b) can occur. This analysis 1374 does not need N to be an integer and indeed shows that the unique solutions to problem 1375

$$\begin{cases} -(t^{M-1}V')' = t^{M-1}V^{p_M} & \text{in } t > 0 \\ V > 0 \end{cases}$$
(2.14)

for M > 2 are the ones in a), b) and c) with N substituted by M. In particular the solutions 1378 in a) are the unique bounded solutions to (2.14) for every $\lambda > 0$. Imposing also the condition 1379

$$V(0) = 1$$
 (2.15)

implies that $\lambda = \sqrt{M(M-2)}$ so that 1382

$$V_M(r) = \left(1 + \frac{r^2}{M(M-2)}\right)^{-\frac{M-2}{2}}$$

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as in (2.16), is the unique bounded solution to (2.14) that satisfies (2.15). 1384

Further we observe that, due the singular behavior at the origin, the solutions b) and c) 1385 do not belong to the space $\mathcal{D}_M(0,\infty)$ which is embedded in $L_M^{p_M+1}(0,\infty)$ for $p_M = \frac{M+2}{M-2}$. 1386 Therefore the solutions in a), for every $\lambda > 0$, are also the only solutions to (2.14) belonging 1387 to $\mathcal{D}_M(0,\infty)$. In particular one sees that every solution in $\mathcal{D}_M(0,\infty)$ also belong to $C[0,\infty)$. 1388 Thus we can also impose the condition (2.15) obtaining that V_M is the unique $\mathcal{D}_M(0,\infty)$ 1389 solution to (2.14) that satisfies (2.15). 1390

The previous discussion applies to the study of radial solutions to 1391

$$\begin{cases} -\Delta U = |x|^{\alpha} U^{p_{\alpha}} & \text{in } \mathbb{R}^{N} \\ U > 0 \end{cases}$$
(5.1)

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where $p_{\alpha} = \frac{N+2+2\alpha}{N-2}$. Indeed, it has been proved in [24] that the transformation 1393 $t = r^{\frac{2+\alpha}{2}}$

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transforms radial $D^{1,2}(\mathbb{R}^N)$ solutions to (5.1) into $\mathcal{D}_M(0,\infty)$ solutions to (2.14) with M as 1395 in (2.4) and M > 2. Performing the previous change of variable into V_M and recalling that 1396 $p_{\alpha} = p_M$ we get that the unique bounded solutions to (2.14) are given by 1397

$$U_{\alpha,\lambda}(x) := \left(\frac{\lambda\sqrt{(N+\alpha)(N-2)}}{\lambda^2 + |x|^{2+\alpha}}\right)^{\frac{N-2}{2+\alpha}}$$

and, imposing the condition 1399

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$$U(0) = 1$$
 (5.2)

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E Journal: 526 Article No.: 1606 TYPESET DISK LE CP Disp.:2019/9/5 Pages: 47 Layout: Small we get that the unique radial bounded solution to (5.1) that satisfies (5.2), i.e. the unique solution to (1.10), is

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$$U_{\alpha}(x) := \left(1 + \frac{|x|^{2+\alpha}}{(N+\alpha)(N-2)}\right)^{-\frac{N-2}{2+\alpha}}$$

as in (1.9). Finally the relation between $D^{1,2}(\mathbb{R}^N)$ and $\mathcal{D}_M(0,\infty)$ also implies that U_{α} is the unique $D^{1,2}(\mathbb{R}^N)$ solution to (5.1) that satisfies (5.2).

Next we look at the generalized radial singular eigenvalue problem associated with the solution V_M , namely

$$-(t^{M-1}\eta')' = t^{M-1}\left(W + \frac{\beta}{r^2}\right)\eta \quad \text{in } t > 0, \tag{3.19}$$

where $W = \frac{M+2}{M-2} \left(1 + \frac{r^2}{M(M-2)}\right)^{-2}$ has been introduced in (3.18), and we look for solutions in $\mathcal{D}_M(0, \infty)$, namely solutions that satisfy

$$\int_0^\infty t^{M-1} \eta' \varphi' \, dt = \int_0^\infty t^{M-1} \left(W + \frac{\beta}{r^2} \right) \eta \varphi$$

1413 for every $\varphi \in C_0^{\infty}(0, +\infty)$.

The generalized radial singular eigenvalue problem (3.19) is of the same type of the previous one (3.4) and indeed the eigenvalues are defined as far as $\beta < \left(\frac{M-2}{2}\right)^2$ and they share the same properties of the previous eigenvalues $\hat{\nu}(p)$. In particular each eigenvalue is simple and the *i*-th eigenfunction admits *i* nodal zones. Then we easily seen that $\beta_1 = -(M-1)$ and $\beta_2 = 0$ with corresponding eigenfunctions

$$\eta_1(r) = \frac{r}{\left(1 + \frac{r^2}{M(M-2)}\right)^{\frac{M}{2}}}, \qquad \eta_2(r) = \frac{1 - \frac{r^2}{M(M-2)}}{\left(1 + \frac{r^2}{M(M-2)}\right)^{\frac{M}{2}}}.$$
(3.20)

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The fact that β_2 is simple implies that $\beta_3 > 0$, so that β_1 and β_2 are the unique non positive eigenvalues of (3.19). See also [24], where the same properties have been used in the proof of Theorem 1.3.

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