

Dynamical analysis of a prey-predator-tourist model: Environmental preferences and optimal fee control

Danilo Delpini ^{*}, Roberta Melis, Paolo Russu

Department of Economics and Business (DISEA), University of Sassari, Via Muroni 25, Sassari, IT-07100, Italy

ARTICLE INFO

Keywords:

Protected area
Bifurcation analysis
Sensitivity analysis
Nonlinear stability analysis
Optimal control
Prey-predator dynamics

ABSTRACT

This paper examines the economic and ecological dynamics that arise in a natural park from the interaction between the tourists that visit the park and the species living there. We consider two species whose interaction is determined by Lotka–Volterra prey–predator equations. The tourists' decision to visit the park is affected by the entrance fee as well as by the possibility of observing the two species, mediated by their preferences towards the prey and predator populations. A third nonlinear equation governs such dynamics. Tourism has conflicting effects: a higher number of visitors can be detrimental to the habitat and its species, but promoting ecotourism while preserving environmental sustainability and equilibrium is also in the mission of protected areas. We analyse the impact on the stability of the equilibrium of different levels of the entrance fee and tourists' preferences for the two species. It is shown that, under specific conditions, an instability may arise leading to species loss and/or no tourists choosing to visit the park. Local and global sensitivity analyses of the equilibrium coordinates with respect to the model inputs highlight the major effects of the entrance fee. Interestingly, the preference for the preys (not the predators) is the crucial parameter when optimizing a fitness utility function for the park in a static setting. Finally, it is shown how to implement an optimal fee-policy control to steer the system towards its stable equilibrium following a path that also maximizes the discounted cumulated utility.

1. Introduction

According to the definition of IUCN, International Union for Conservation of Nature [1], “A protected area is a clearly defined geographical space, recognized, dedicated and managed, through legal or other effective means, to achieve the long term conservation of nature with associated ecosystem services and cultural values”. Protected Areas (PAs) have to be managed to fulfil two main objectives: preserving natural environments and landscapes as well as promoting recreational and cultural activities [2]. However, the management and maintenance of these areas require significant financial resources.

An entrance fee can offer a solution to these financial needs, helping to cover the costs of management, monitoring and protection of the areas, as well as promoting responsible tourism. If well-structured, an entrance fee can also serve as an educational tool, raising visitors' awareness of the importance of conservation and encouraging environmentally respectful behaviour. However, it is crucial that the fee amount and application methods are carefully balanced to ensure that PAs remain accessible to different population groups without compromising their conservation value.

Eagles et al. [3] in their book provide a comprehensive guide on planning and managing sustainable tourism in PAs, including a discussion on the importance of entrance fees to support the management and conservation of natural resources. Buckley [4] explores the political and managerial implications of introducing entrance fees in public parks in Australia, analysing the effects on visitors and conservation. Drumm [5] explores how entrance fees and tourism concessions can create a financial threshold of sustainability for PAs. The paper emphasizes balancing ecological protection with tourism development to ensure long-term conservation funding. Yoon and Zou [6] make use of the zero-price effect and transactional utility theory. They investigate the impact of entrance fees on the experiences and decisions of park visitors by targeting different populations with experiments based on written vignettes. Lupi et al. [7] analyse the profit generated via uniform entrance fees across various locations in comparison to the extra state income tax that produces equal revenues. Total trip and site needs exhibit cost elasticity over a wide spectrum of entrance rates. In Gao et al. [8] the Random Regret Minimization (RRM) model is introduced. They consider polynomial logit and hybrid latent class models and examine their capability to analyse discrete choice models based on both utility and regret.

^{*} Corresponding author.

E-mail address: ddelpini@uniss.it (D. Delpini).

Many authors studying the impact of tourists/visitors have incorporated entrance fees into their mathematical models. Sarkar et al. [9] propose an ecological-economic predatory-prey fishery model with the objective of demonstrating the roles of fishing tax and tourist entrance fees in stabilizing fishery dynamics and maximizing revenue generation. The dynamic evolution is studied through the investigation of the different equilibrium points and bifurcation analysis. Notaro and Grilli [10] conduct a discrete choice experiment to assess tourist preferences for the conservation of large carnivores in the Italian Alps. Russu [11] presents a mathematical model that takes into account the number of visitors, the quality of the environment, and the existing infrastructure, focusing on the use of tourism fees to fund environmental protection and improve facilities. Through stability and bifurcation analysis, it is demonstrated that delays in the implementation of protection measures can lead to instability in the system, with negative consequences for the sustainable management of PAs. The author concludes that adequate and timely investments in environmental protection are essential to mitigate the negative impact of tourism and ensure the long-term sustainability of these areas.

Visitor preferences regarding animals in PAs are a fundamental aspect of managing nature-based tourism and biodiversity conservation. Charismatic animals, such as large mammals (e.g., elephants, lions, and bears), often capture the attention of visitors and can significantly influence their travel decisions. This phenomenon, known as “wildlife charisma”, not only shapes tourists’ experiences but can also determine conservation priorities within PAs.

Understanding visitor preferences for specific species or animal groups is essential for developing effective management strategies that balance tourist interest with the need to protect biodiversity. Preferences can vary widely based on cultural, demographic and personal factors, and analysing these preferences can provide valuable insights into how to enhance visitor experiences, promote environmental education, and ensure that conservation efforts are not limited to the most popular species.

However, excessive focus on certain species could lead to unbalanced management, where resources and conservation efforts are concentrated only on a few species at the expense of others that are less visible but equally important for the ecosystem. Therefore, a thorough analysis of visitor preferences can help guide more inclusive and sustainable management decisions in PAs.

Numerous studies have explored tourist preferences for wildlife, providing estimates of the preferred species by completing a questionnaire. Kerley et al. [12] suggest that tourists’ preference for charismatic mega-fauna may lead to a reduced appreciation for biodiversity. Lindsey et al. [13] investigate, by completing questionnaires, tourist preferences in four PAs in South Africa, finding that that mega-herbivores and large carnivores were favoured by visitors, especially first-time and international ones. Conversely, local African visitors and seasoned wildlife enthusiasts showed more interest in bird and plant diversity, scenic views, and rare, less easily spotted mammals. Di Minin et al. [14] study heterogeneous preferences of tourists for big game species in South Africa, identifying two segments defined largely by socio-economic characteristics. Maciejewski and Kerley [15] examine tourist preferences for mammal species in a private PA in South Africa revealing a strong preference for larger and more charismatic species such as elephants, lions, leopards, and cheetahs.

Guo and Fennel [16] rely on consumer learning theory and lively capital to further explain the dynamic relationship existing between preference for animals and visitation experiences. Obrodovic et al. [17] explore how encounters with keystone species or flagship species can significantly impact tourist satisfaction and influence visitation patterns in nature-based tourism settings. Wallace [18] analyses how visitor preferences for charismatic or rare species can influence conservation strategies in national parks, as well as the role of these species in attracting tourists, thereby benefiting both conservation funding and local economies. Mzek et al. [19] study the population of tigers,

availability of tiger prey, presence of rangers, frequency of awareness programmes, buffer zones, and conservation fees. The findings indicate that education and gender substantially influenced visitors’ choices. Individuals with advanced education are inclined to invest in the enhancement of tiger prey populations. Eyster et al. [20] try to single out ecotourist preferences and to assess which species and landscapes benefit the most from ecotourism, both aspects being relevant to drive efforts to preserve African biodiversity.

While the role of entrance fees has been widely investigated, the literature does not provide dynamic models that incorporate tourists’ preferences for different animal species in the PAs. So, our objective is to analyse a stylized model of the interactions between tourists and two species — prey and predator — within a natural PA setting, governed by a system of nonlinear differential equations. We conduct such analysis in terms of equilibrium points and bifurcations, as well as of sensitivity to model inputs. Furthermore, we intend to study the dynamics subject to optimal control, where the entrance fee represents the control variable.

This work contributes to the growing body of literature on sustainable tourism management by proposing an optimal fee policy that balances ecological preservation and economic gain. By offering insights into how different variables interact to shape the dynamics of species populations and visitors, we provide a framework for managing entrance costs consistently with tourists’ preferences, in a way that maximizes long-term benefits for both conservation of the species and tourism fostering. However, it is not the aim of this work to build a completely realistic model capturing all aspects, which would come at the cost of losing any analytical tractability. Nor we investigate here the problem of building a differential model informed by data. That is clearly a crucial issue and is tackled by many empirical articles that adopt a data driven approach, such as recent studies using neural networks to model time series. For instance, Altan et al. [21] develop a hybrid forecasting model based on long short-term memory neural network and empirical wavelet transform decomposition along with cuckoo search algorithm for digital currency time series, showing that the combined model can capture nonlinear properties of digital currency time series. Pekkaya et al. [22] employ artificial neural networks to identify the main factors affecting sales volume in an iron and steel company, providing a more accurate prediction than classical regression models.

The work is organized as follows: Section 2 describes the model in a prey-predator setting; Section 3 analyses the existence and stability of equilibria; in Section 4 a sensitivity analysis is performed at the interior equilibrium point with respect to the entrance fee and the tourists’ preferences for the two species; in Section 5 an optimization on the entrance fee is performed; Section 6 concludes the paper.

2. The model

The classical Lotka–Volterra model [23,24] defines a fundamental mathematical framework for studying predator–prey interactions. In ecology, innumerable studies on interacting populations have been based on varying formulations of the model (see among the others [25–29]). Following, e.g. [30], we consider the following dynamical system

$$\begin{aligned}\dot{x} &= F(x, y; \mathcal{P}_x) = ax - bx^2 - cxy \\ \dot{y} &= G(x, y; \mathcal{P}_y) = -dy + eyx - iy^2\end{aligned}\quad (1)$$

where $\mathcal{P}_x \equiv \{a, b, c\}$ and $\mathcal{P}_y \equiv \{d, e, i\}$ are sets of parameters.¹ In (1) the terms $ax - bx^2$ define a logistic growth with a limiting carrying capacity

¹ The literature contains several theoretical analyses [31–33] of how \mathcal{P}_x and \mathcal{P}_y determine the dynamic trajectory of the state variables. It has to be mentioned that it is difficult to obtain adequate data and historical series and we remark that our model is not based on a data-driven approach. Nevertheless, we have supported the model with references to relevant empirical studies and experiments on the subject.

a/b (see e.g., [34]), and $cx y$ represents loss in prey biomass due to predation (also known as the functional response [35]). The predator dynamics is determined by a natural death rate term $d y$, population decrease due to intraspecific competition $i y^2$, and $ex y$, which defines the biomass gain through predation. Model (1) provides a commonly accepted, albeit simplified, framework for the description of competing species that remains analytically tractable.

In contexts such as large national parks, the ample space and available resources for predators may minimize the need for intraspecific competition. With sufficient prey and spacious territories, predators are less likely to compete significantly, making intraspecific competition less relevant. Moreover, many national parks monitor and sometimes control predator numbers to prevent excessive impact on overall wildlife and prey species. This helps maintain predator populations at stable levels, preventing them from reaching densities that would increase intraspecific competition. Given these considerations, we assume intraspecific competition to be negligible and the parameter i to be zero, making system (1) become

$$\begin{aligned} \dot{x} &= x(a - bx - cy) \\ \dot{y} &= y(-d + ex) \end{aligned} \tag{2}$$

which we call the “natural system”, where we assume that all the parameters are non-negative and that the quantity $q_N := ae - bd$ is strictly positive. Under these assumptions, system (2) has a positive stationary solution with coordinates $(x_N, y_N) = (d/e, q_N/ec)$.

Now let us introduce the possibility for the tourists to visit the park. Tourists are often drawn to protected areas for a variety of reasons, primarily including: nature immersion; adventure and outdoor activities; wildlife observation; cultural and historical interest; sustainable tourism. The main obstacles discouraging tourists from visiting protected areas include: limited accessibility; high costs; lack of infrastructure; perceived risk.

The introduction of tourists into the model leads to a modification of the natural dynamics (2). In this work we are interested primarily in modelling the dynamics of tourists as influenced by the factors “Wildlife Observation” and “High Costs” with reference to the entrance fee. If T is the density of tourist population, the rate of variation per unit of T will be a function of x, y and T , depending on a number of other parameters, that is $\dot{T}/T = F(x, y, T; \dots)$. We will assume F to be linear in x, y, T .

Thus we propose the following three-dimensional model where we added the third equation describing the dynamic evolution of tourists visiting the park:

$$\begin{cases} \dot{x} = x f(x, y, T) = x(a - bx - cy - \alpha_1 T^\gamma) \\ \dot{y} = y g(x, y, T) = y(-d + ex - \alpha_2 T^\gamma) \\ \dot{T} = T h(x, y, T) = T(-\beta + \sigma_1 x + \sigma_2 y) \end{cases} \tag{3}$$

System (3) is defined on the set $\Gamma = \{(x, y, T) \in \mathbb{R}^3 | x \geq 0, y \geq 0, T \geq 0\}$, subject to the initial conditions

$$x(0) > 0, y(0) > 0, T(0) > 0 \tag{4}$$

The term $-\beta$ is a loss term that is responsible for slowing down the growth of the tourist population. That could arise from an increase in entrance fees or other costs associated with visiting the park. Indeed, it is reasonable to expect that an increase in costs (for example, due to higher entrance fees) leads to a decrease in tourist demand and negatively affects the number of visitors per unit time, which is supported by [36]. For the sake of brevity, in the following we will refer to the parameter β simply as the “entrance fee”.

The parameters σ_1 and σ_2 are related to the preferences of tourists for the prey and predator species respectively. Tourists may be attracted by the presence of preys, which offers opportunities for wildlife observation and recreational activities such as birdwatching or photographic safaris. An increase in the prey population can therefore stimulate an

increase in the number of tourists, which is quantified by the term $\sigma_1 x$ and is proportional to the number of potential encounters per tourist with the preys (x). Similarly, the presence of predators may attract tourists interested in unique and exciting observation experiences. Therefore, an increase in the predator population may also contribute to an increase in the number of tourists, as indicated by the term $\sigma_2 y$, again proportional to the number of potential encounters per tourist with the predators (y). Such kind of effects are also supported by previous empirical studies, see e.g. [13,14].

The presence of the tourists entails a disturbance to the species. While much research has focused on the effects of human disturbance on wildlife behaviour, a growing body of evidence suggests that fear of the human “super predator” may often be the ultimate driver of wildlife responses to such disturbance. Recent meta-analyses of worldwide data highlight alterations in movement [37] and increases in nocturnality [38] of terrestrial mammals of all sizes and types in response to human disturbance. Some experiments have indicated that the fear instilled in large and medium carnivores by the human “super predator” can affect how frequently large carnivores kill preys [39] and induce trophic cascades² [40,41].

In terms of the rates of variation per individual, \dot{x}/x and \dot{y}/y , we model such effects by the nonlinear terms $-\alpha_1 T^\gamma$ and $-\alpha_2 T^\gamma$, with $\gamma > 1$. The parameters α_1 and α_2 modulate the disturbance from the tourists on the prey and predator population respectively. There exist ecological and behavioural reasons supporting the idea that the amount of disturbance, per individual of the species, should be nonlinear in the number of disturbers [42–44]. For instance, threshold effects can be observed: disturbance may remain minimal up to a certain level of tourist presence and then increase rapidly beyond a critical threshold. The dependence on tourist density is another factor: the impact of the disturbance grows with the local density of tourists, affecting the species unevenly. Also, predators and preys may develop different tolerances and adaptations to human presence, generating differentiated nonlinear responses. Finally, cumulative stress may play an important role: the effects of disturbance tend to accumulate over time, progressively reducing the survival and reproductive capacities of the species disturbed. For the sake of simplicity, here we assume $\gamma = 2$ for the exponent of the nonlinear disturbance.

In ecological modelling, positivity and boundedness ensure that the populations of species remain realistic over time, no species can have a negative population, and no population can grow without limit. In what follows, we demonstrate that the solutions of our model are positive and bounded if the parameters satisfy certain inequalities.

Lemma 1. *The solutions $(x(t), y(t), T(t))$ of system (3) under the initial conditions (4) are (a) positive for all $t \geq 0$ and (b) uniformly bounded if $c > e$.*

Proof.

(a) Eqs. (3) together with the conditions (4) give

$$x(t) = x(0) \exp\left(\int_0^t f(x(s), y(s), T(s)) ds\right) > 0$$

$$y(t) = y(0) \exp\left(\int_0^t g(x(s), y(s), T(s)) ds\right) > 0,$$

$$T(t) = T(0) \exp\left(\int_0^t h(x(s), y(s), T(s)) ds\right) > 0.$$

Hence all solutions starting from an interior point of the first octant remain in it at all future times.

² Trophic cascades are strong indirect interactions that can govern entire ecosystems. They occur when predators limit the density and/or behaviour of their prey and thereby enhance survival of the next lower trophic level.

(b) Let us consider $W(t) = x(t) + y(t) + mT(t)^2$, then the time derivative along the solutions reads

$$\frac{dW}{dt} = x(t)(a - bx(t) - cy(t) - \alpha_1 T(t)^2) + y(t)(-d + ex(t) - \alpha_2 T(t)^2) + 2mT(t)^2(-\beta + \sigma_1 x(t) + \sigma_2 y(t))$$

We can rewrite the above equation as

$$\frac{dW}{dt} + \eta W = bx(t) \left(\frac{a + \eta}{b} - x(t) \right) - x(t)y(t)(c - e) - (d - \eta)y(t) - (\alpha_1 - 2m\sigma_1)x(t)T(t)^2 + - (\alpha_2 - 2m\sigma_2)y(t)T(t)^2 - (2\beta - \eta)mT(t)^2.$$

where η is a positive constant. We consider $m = \min \left\{ \frac{\alpha_2}{2\sigma_2}, \frac{\alpha_1}{2\sigma_1} \right\}$ and $\eta = \min\{d, 2\beta\}$, also we consider $c > e$, then we have

$$\frac{dW}{dt} + \eta W \leq -bx(t) - \frac{a + \eta}{2b} + \frac{(a + \eta)^2}{4b^2}$$

which gives

$$\frac{dW}{dt} + \eta W \leq \frac{(a + \eta)^2}{4b^2}$$

Applying the theory of differential inequality, we obtain

$$0 < W(t) \leq \frac{(a + \eta)^2}{4b^2\eta} (1 - e^{-\eta t}) + W(0)e^{-\eta t}.$$

Hence

$$0 < \lim_{t \rightarrow +\infty} W(t) \leq \frac{(a + \eta)^2}{4b^2\eta}.$$

So all solutions of system (3) starting from \mathbb{R}^3 are confined in the region

$$\Omega = \{(x, y, T) \in \mathbb{R}^3 : x(t) + y(t) + mT(t)^2 < \frac{(a + \eta)^2}{4b^2\eta} + \phi,$$

for any $\phi > 0\}$ ■

Model (3) provides an admittedly simplified representation of the real-world interactions between tourists and animal species. Many important effects, such as seasonality or tourism pressure on the environment, are not included here. The equation for \dot{T}/T is linear but also coherent in the spirit with the classic Lotka–Volterra equations, being the sum of a constant which would be responsible for an exponential decrease of T , with cross terms xT and yT representing prey–tourist and predator–tourist interactions in terms of number of encounters. The model provides a reasonable stylized description of basic facts regarding those interactions. It does so retaining a considerable degree of analytical tractability, which allows for a thorough analysis of the dynamical system and its equilibria, as discussed in the following sections.

It is also to be emphasized that we will not consider any aspects relating to revenues and profits from managing the park, even though β is understood as an entrance fee. It is implied that the revenue βT covers the costs of running the park and sustains actions of environmental and wildlife preservation that are distinctive of PAs. Rather, we will treat β as a control parameter. On the one hand, we will show in the next Section that it is essential to preserve a stable equilibrium in the system. Ideally it is to be fixed based on preliminary estimation of the tourists’ preferences. On the other, it will also be instrumental in driving the system towards the equilibrium over a path that maximizes utility, see Section 5.2. In doing so, β will also allow to estimate a lower bound on the time to equilibrium.

3. Existence and stability of equilibria

We now consider the conditions for the existence and local asymptotic stability of the equilibrium points, and for the persistence of the solutions. We also study the instability of the interior equilibrium through a Hopf bifurcation. If not otherwise specified, we will always

assume $q_N = ae - bd > 0$, so guaranteeing that a natural equilibrium between the species can be attained without any tourists.

Proposition 1. *The equilibria of system (3) are the following.*

(i) *The trivial equilibrium $P_0 = (0, 0, 0)$ and the axial equilibrium $P_x = \left(\frac{a}{b}, 0, 0\right)$ exist irrespective of any parametric restrictions.*

(ii) *If $q_N > 0$ a boundary equilibrium point in the xy -plane exists with the coordinates $P_{xy} = (x_N, y_N, 0)$.*

(iii) *Under the parametric restriction $\sigma_1 > \beta b/a$, or equivalently $\beta < \sigma_1 a/b$, a boundary equilibrium point in the xT -plane exists with the coordinates*

$$P_{xT} = \left(\frac{\beta}{\sigma_1}, 0, \sqrt{\frac{\sigma_1 a - \beta b}{\alpha_1 \sigma_1}} \right) \tag{5}$$

(iv) *A unique equilibrium point P_{xyT} in the interior may exist with the coordinates*

$$x_I = \frac{d + \alpha_2 T_I^2}{e} = x_N + \frac{\alpha_2}{e} T_I^2, \quad y_I = y_N - AT_I^2, \quad T_I = \sqrt{\frac{y_N - B}{A - C}}, \tag{6}$$

where $A = (\alpha_2 b + \alpha_1 e)/ce > 0$, $B = (\beta e - \sigma_1 d)/\sigma_2 e$, $C = \alpha_2 \sigma_1/\sigma_2 e > 0$. Such equilibrium exists if the following parametric restrictions are both satisfied

$$\sigma_2 > \hat{\sigma}_2(\beta, \sigma_1) := \frac{c(\beta e - \sigma_1 d)}{q_N}, \quad \sigma_1 < \bar{\sigma}_1(\beta) := \frac{\beta(\alpha_1 e + \alpha_2 b)}{\alpha_1 d + \alpha_2 a} \tag{7}$$

or if they hold with the inequalities reversed.

Proof. Claims (i)–(iii) can be verified easily. To obtain the coordinates of the interior equilibrium point, we put $f(x, y, T) = g(x, y, T) = h(x, y, T) = 0$. From $h = 0$ we obtain $x = (\beta - \sigma_2 y)/\sigma_1$. Replacing in $f = 0$ and $g = 0$ and by algebraic manipulation, we obtain the coordinates (6). Those coordinates are feasible as long as

$$\begin{cases} \frac{y_N - B}{A - C} > 0 \\ y_N > AT_I^2 \end{cases}$$

Introducing the function $\bar{\sigma}_2(\sigma_1) = \alpha_2 \sigma_1 c / (\alpha_1 e + \alpha_2 b)$, the first condition of the system requires

$$\sigma_2 > \max \{ \hat{\sigma}_2, \bar{\sigma}_2 \} \quad \vee \quad \sigma_2 < \min \{ \hat{\sigma}_2, \bar{\sigma}_2 \}$$

Since $A > 0$, the second condition of the system requires

$$(\sigma_2 > \bar{\sigma}_2, \sigma_1 < \bar{\sigma}_1) \quad \vee \quad (\sigma_2 < \bar{\sigma}_2, \sigma_1 > \bar{\sigma}_1)$$

We conclude that the system is satisfied if

$$(\sigma_2 > \max \{ \hat{\sigma}_2, \bar{\sigma}_2 \}, \sigma_1 < \bar{\sigma}_1) \quad \vee \quad (\sigma_2 < \min \{ \hat{\sigma}_2, \bar{\sigma}_2 \}, \sigma_1 > \bar{\sigma}_1)$$

For a fixed β the graphs of $\hat{\sigma}_2(\beta, \sigma_1)$, $\bar{\sigma}_2(\sigma_1)$ and $\bar{\sigma}_1(\beta)$ are straight lines in the $\sigma_1 \sigma_2$ -plane. It can easily be verified that $\hat{\sigma}_2 = \bar{\sigma}_2$ at exactly $\sigma_1 = \bar{\sigma}_1$. Moreover, for $q_N > 0$ we have $\hat{\sigma}_2 > \bar{\sigma}_2$ for $\sigma_1 < \bar{\sigma}_1$, and $\hat{\sigma}_2 < \bar{\sigma}_2$ for $\sigma_1 > \bar{\sigma}_1$. In the light of that, for fixed β we can state the existence conditions for P_{xyT} more simply as

$$(\sigma_2 > \hat{\sigma}_2, \sigma_1 < \bar{\sigma}_1) \quad \vee \quad (\sigma_2 < \hat{\sigma}_2, \sigma_1 > \bar{\sigma}_1) \tag{8}$$

which proves claim (iv). ■

The coordinates of P_{xyT} can be rewritten conveniently as

$$\begin{cases} x_I = \frac{1}{e} \left(d + \frac{\alpha_2 q_N}{\alpha_1 e + \alpha_2 b} \cdot \frac{\sigma_2 - \hat{\sigma}_2}{\sigma_2 - \bar{\sigma}_2} \right) \\ y_I = -\frac{\alpha_1 d + \alpha_2 a}{\alpha_1 e + \alpha_2 b} \cdot \frac{\sigma_1 - \bar{\sigma}_1}{\sigma_2 - \bar{\sigma}_2} \\ T_I = \sqrt{\frac{q_N}{\alpha_1 e + \alpha_2 b} \cdot \frac{\sigma_2 - \hat{\sigma}_2}{\sigma_2 - \bar{\sigma}_2}} \end{cases} \tag{9}$$

These expressions have the advantage of making the conditions (8) explicit. The conditions can also be rephrased depending on which variable we consider fixed. If we introduce the function $\bar{\beta}(\sigma_1) = (\alpha_1 d + \alpha_2 a) \sigma_1 / (\alpha_1 e + \alpha_2 b)$, then it is $\sigma_1 < \bar{\sigma}_1$ if and only if $\beta > \bar{\beta}$ and the conditions become

$$(\beta > \bar{\beta}, \sigma_2 > \hat{\sigma}_2) \vee (\beta < \bar{\beta}, \sigma_2 < \hat{\sigma}_2) \tag{10}$$

Finally, focusing on the $\beta\sigma_1$ -plane for a fixed σ_2 , we can define $\hat{\sigma}_1(\beta, \sigma_2) = \beta e/d - \sigma_2 q_N/cd$ and observe that $\sigma_2 > \hat{\sigma}_2$ if and only if $\sigma_1 > \hat{\sigma}_1$. Then the feasibility conditions for P_{xyT} in the $\beta\sigma_1$ -plane can be rephrased as

$$\hat{\sigma}_1 < \sigma_1 < \bar{\sigma}_1 \vee \bar{\sigma}_1 < \sigma_1 < \hat{\sigma}_1 \tag{11}$$

In the following proposition we provide local stability conditions for the feasible equilibrium points of system (3), under the assumption $q_N > 0$, based on linearization and the Routh–Hurwitz criterion.

Proposition 2. *The stability of the equilibria in Proposition 1, as deduced from the eigenvalues of the Jacobian matrix $J(x, y, T)$, is summarized as follows.*

- *Stability of P_0*
The eigenvalues of $J(P_0)$ are $\lambda_1 = -d$, $\lambda_2 = -\beta$, $\lambda_3 = a$, hence the origin is always a saddle with two-dimensional stable manifold.
- *Stability of P_x*
 $J(P_x)$ has the eigenvalues $\lambda_1 = -a$, $\lambda_2 = q_N/b$ and $\lambda_3 = (\sigma_1 a - \beta b)/b$. The axial equilibrium point P_x is a saddle with two-dimensional stable manifold if $\sigma_1 < \beta b/a$, otherwise it is a saddle with one-dimensional stable manifold.

- *Stability of P_{xy}*
The eigenvalues of $J(P_{xy})$ read
$$\lambda_{1,2} = -\frac{bd}{2e} \left(1 \mp \sqrt{\Delta} \right), \quad \lambda_3 = \frac{q_N}{ce} (\sigma_2 - \hat{\sigma}_2)$$
where $\Delta = 1 - 4eq_N/(b^2 \cdot d)$. Since $q_N > 0$ we have $\Delta < 1$ and $\text{Re}(\lambda_{1,2}) < 0$. So P_{xy} is a locally stable equilibrium if $\sigma_2 < \hat{\sigma}_2$ (or $\sigma_1 < \hat{\sigma}_1$ equivalently) otherwise it is a saddle with two-dimensional stable manifold. In particular, if $\sigma_2 < \hat{\sigma}_2$ and $0 \leq \Delta < 1$ the equilibrium is a stable node.

- *Stability of P_{xT}*
The eigenvalues of $J(P_{xT})$ read
$$\lambda_{1,2} = -\frac{\beta b}{2\sigma_1} \left(1 \mp \sqrt{\Delta} \right), \quad \lambda_3 = -\frac{\alpha_1 d + \alpha_2 a}{\alpha_1 \sigma_1} (\sigma_1 - \bar{\sigma}_1)$$
where $\Delta = 1 - 8\sigma_1(\sigma_1 a - \beta b)/\beta b^2$. Since $\sigma_1 > \beta b/a$ for P_{xT} to exist, we have $\Delta < 1$ and $\text{Re}(\lambda_{1,2}) < 0$. If $\sigma_1 > \bar{\sigma}_1$ (or $\beta < \bar{\beta}$ equivalently) the point is a locally stable equilibrium (a stable node if $0 \leq \Delta < 1$), otherwise it is a saddle with two-dimensional stable manifold.

- *Stability of P_{xyT}*
The Jacobian evaluated at the interior equilibrium point is
$$J_{xyT} := J(P_{xyT}) = \begin{pmatrix} -bx_I & -cx_I & -2\alpha_1 x_I T_I \\ ey_I & 0 & -2\alpha_2 y_I T_I \\ \sigma_1 T_I & \sigma_2 T_I & 0 \end{pmatrix} \tag{12}$$

The characteristic equation associated with J_{xyT} reads $\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$, where $a_2 = bx_I$, $a_1 = cex_I y_I + 2T_I^2(\alpha_1 \sigma_1 x_I + \alpha_2 \sigma_2 y_I)$ and $a_0 = 2x_I y_I T_I^2(\alpha_2 \sigma_2 b + \alpha_1 \sigma_2 e - \alpha_2 \sigma_1 c)$. First we observe that $a_1 > 0$ and $a_2 > 0$ and that a_0 can be written as

$$a_0 = 2q_N(\sigma_2 - \hat{\sigma}_2)x_I y_I$$

Then, by virtue of the Routh–Hurwitz criterion, P_{xyT} is locally asymptotically stable if and only if $\sigma_2 > \hat{\sigma}_2$ and $a_2 a_1 > a_0$. If that is the case, by Descartes’ rule P_{xyT} may be a stable node, for three negative eigenvalues, or a stable equilibrium with one negative eigenvalue plus two complex ones in the open left half-plane. If $\sigma_2 > \hat{\sigma}_2$ but $a_2 a_1 \leq a_0$, then J_{xyT} must have one negative eigenvalue and two complex ones with non negative real part, which would make P_{xyT} a saddle with one-dimensional stable manifold. Finally, if $\sigma_2 < \hat{\sigma}_2$

then J_{xyT} has one positive eigenvalue. It may have either two further eigenvalues with negative real parts, corresponding to a saddle with two-dimensional stable manifold, or two further complex eigenvalues in the right half-plane, corresponding to an unstable equilibrium with no stable manifold.

Remark 1. Note that the conditions $\sigma_1 < \bar{\sigma}_1$ and $\sigma_2 > \hat{\sigma}_2$ are necessary and sufficient to ensure that the equilibrium P_{xyT} exists and it is locally asymptotically stable or a saddle with one-dimensional stable manifold, depending on the sign of $a_2 a_1 - a_0$. The conditions $\sigma_1 > \bar{\sigma}_1$ and $\sigma_2 < \hat{\sigma}_2$, on the other hand, guarantees the existence of an unstable interior equilibrium with two-dimensional stable manifold or no stable manifold at all. Rewriting $\sigma_2 > \hat{\sigma}_2$ in terms of β , the following condition is necessary for P_{xyT} to exist as an attractor

$$\bar{\beta} = \frac{(\alpha_1 d + \alpha_2 a) \sigma_1}{\alpha_1 e + \alpha_2 b} < \beta < \frac{d}{e} \cdot \sigma_1 + \frac{q_N}{ce} \cdot \sigma_2 = \beta_{\text{sup}} \tag{13}$$

Remark 2. Note that the equilibrium points P_{xy} and P_{xT} are globally asymptotically stable in the interior of the positive quadrant of the xy -plane and xT -plane, respectively. With regard to the planar point P_{xy} , we consider the following functions $H(x, y) = \frac{1}{xy}$, $f(x, y, 0) = f_1 = x(a - bx - cy)$ and $g(x, y, 0) = g_1 = y(-d + ex)$. Clearly, $H > 0$ in the interior of the positive quadrant of the xy -plane. Thus we have

$$\Delta(x, y) = \frac{\partial}{\partial x}(f_1 H) + \frac{\partial}{\partial y}(g_1 H) = -\frac{b}{y} < 0.$$

The quantity $\Delta(x, y)$ has constant sign and is not identically zero in the positive quadrant of the xy -plane. Therefore, by Bendixson–Dulac criterion, P_{xy} is globally asymptotically stable in the interior of the positive quadrant of the xy -plane. The claim regarding the planar point P_{xT} can be proven in a similar way.

Table 1 summarizes the conditions for the existence and stability of the equilibria of model (3), together with their classification. In Fig. 1 we show the equilibria (stable or unstable) in the $\sigma_2\sigma_1$, $\sigma_2\beta$ - and $\sigma_1\beta$ -plane, for the following benchmark values of those parameters that are not running.

$$\begin{aligned} a = 5, b = 0.25, c = 1, d = 1, e = 0.5, \\ \alpha_1 = 0.00015, \alpha_2 = 0.0001, \\ \beta = 8, \sigma_1 = 0.7, \sigma_2 = 1.8, \gamma = 2 \end{aligned} \tag{14}$$

A quantitative approach to the management of PAs should take into account the effects of the variations of $\beta, \sigma_1, \sigma_2$ on the variables x, y, T at the equilibrium. Such matter will be discussed in more detail in Section 4. Here we just comment on the effects of changes to β and the preferences on the stable equilibria of the system, assuming that initially it is close to the interior equilibrium.

First we consider β fixed. For σ_1 also fixed, when σ_2 becomes smaller than $\hat{\sigma}_2$ then P_{xy} is the only stable equilibrium: the preference for the predators is then crucial in maintaining tourism. Conversely, for σ_2 fixed, when $\sigma_1 > \bar{\sigma}_1$ the point P_{xT} is an attractor: a strong preference towards the preys leads to the extinction of the predators. When the previous conditions are met together, P_{xy} and P_{xT} coexist as stable equilibria. At the opposite end, a large reduction in σ_1 or large increase in σ_2 generally correspond to the system crossing the Hopf bifurcation curve and entering a region where P_{xyT} only exists as an unstable equilibrium.

Now, for σ_1, σ_2 both fixed, let us consider the effects of changing β . For P_{xyT} to be an attractor, $\bar{\beta} < \beta < \beta_{\text{sup}}$ must hold at the interior equilibrium P_{xyT} , see Eq. (13). This range sets the boundaries the management of the park should keep to when making changes to the entrance fee. It should be observed that the lower bound $\bar{\beta}$ depends (linearly) on the preference towards the preys alone, while the upper

Table 1

Conditions for the existence and stability of the equilibria of system (3) and their classification, under the assumption $q_N = ae - bd > 0$. The phrases “saddle 2D” and “saddle 1D” mean a saddle with two- or one-dimensional stable manifold respectively. a_2, a_1 and a_0 stand for the coefficients of the characteristic polynomial of the Jacobian $J(P_{xyT})$.

	Existence	Stability	Stable classification	Unstable classification
P_0	Always	Always unstable	–	Saddle 2D
P_x	Always	Always unstable	–	Saddle 2D if $\sigma_1 < \beta b/a$ Saddle 1D otherwise
P_{xy}	Always	$\sigma_2 < \hat{\sigma}_2$	Stable node if: $0 < q_N \leq b^2 d/4e$ Other stable equilibrium if: $q_N > b^2 d/4e$	Saddle 2D
P_{xT}	$\sigma_1 > \beta b/a$	$\sigma_1 > \bar{\sigma}_1$	Stable node if: $8\sigma_1^2 a/(8\sigma_1 b + b^2) \leq \beta < \sigma_1 a/b$ Other stable equilibrium if: $\beta < 8\sigma_1^2 a/(8\sigma_1 b + b^2)$	Saddle 2D
P_{xyT}	$\sigma_2 > \hat{\sigma}_2, \sigma_1 < \bar{\sigma}_1 \vee$ $\sigma_2 < \hat{\sigma}_2, \sigma_1 > \bar{\sigma}_1$	$\sigma_2 > \bar{\sigma}_2 \wedge a_2 a_1 > a_0$	Stable node or other stable equilibrium	Saddle 1D if: $\sigma_2 > \bar{\sigma}_2, a_2 a_1 < a_0$ Saddle 2D or no stable manifold if: $\sigma_2 < \bar{\sigma}_2$

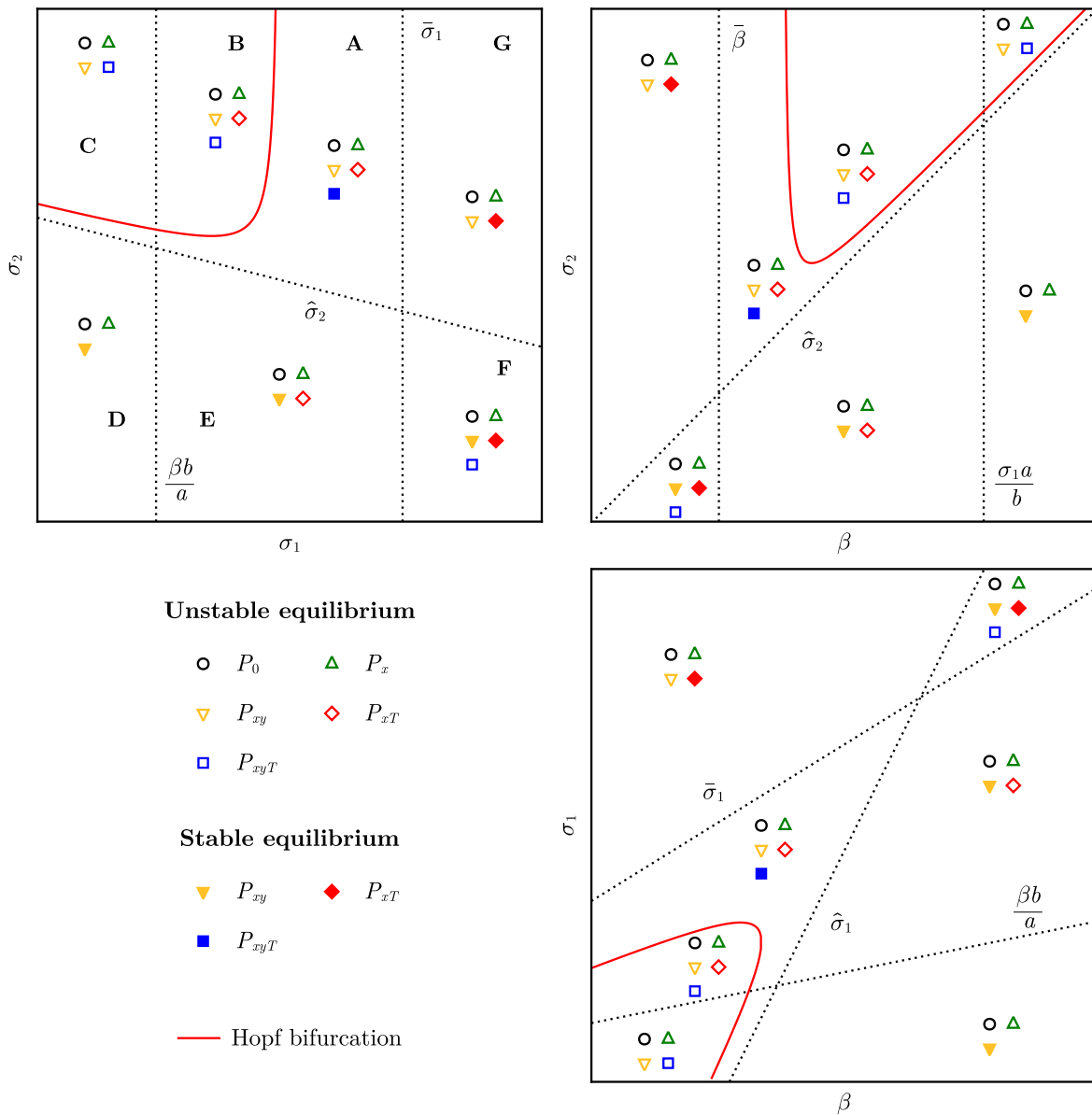


Fig. 1. Existence and stability of the fixed points for the benchmark parameters (14). The top-left panel also classifies seven regions of interest with letters.

bound β_{sup} is given by a linear combination of both preferences. Raising β above β_{sup} eventually leads the system into the region of the attractor P_{xy} (suppression of tourism), while if β goes below $\bar{\beta}$ the extinction of the predators is expected. In both cases, the system may transiently go across the instability region delimited by the Hopf curve.

Such analysis highlights that the admissible values of β are strictly constrained by the tourists' preferences for the species. For values outside the range, species loss or tourism depletion are expected. The latter circumstance arises when $\beta > \beta_{\text{sup}}$: increasing β higher than the upper bound determined by the preferences suppresses tourism. On the other hand, a lower bound for β exists that is determined by σ_1 alone. For $\beta < \bar{\beta}$ more tourists will visit the park, which means more disturbance to the species. Both the predators and the preys would diminish due to the disturbance, but so more the predators since they could prey on less. Net of the two effects, this would ultimately result in the extinction of the predators. Fig. 2 shows some sample trajectories for different choices of σ_1, σ_2 , corresponding to the regions distinguished by capital letters in the top-left panel of Fig. 1.

Such analysis and the above discussion suggest applicable policies that park managers could consider. For instance, let us assume that, for given parameters ($\beta^0, \sigma_1^0, \sigma_2^0$), the system currently occupies region A of the plane, where a stable equilibrium is achieved in the long run with strictly positive $x(\infty), y(\infty), T(\infty)$. In response to changes in the tourists' preferences for the species (over which managers have no direct control), the decision-maker may need to adjust the entrance fee (β). That could avoid shifting to one of the other regions, which would cause the internal equilibrium to become unstable (region B) or lead the protected area to lose its sustainability due to the extinction of one of the species or the "extinction" of the tourists.

Remark 3. Throughout the rest of the article, the values of $a, b, c, d, e, \alpha_1, \alpha_2, \gamma$ are kept fixed at their benchmark values (14). Depending on the analysis, the values of β, σ_1 and σ_2 may vary and their values will be explicitly given. Otherwise it is assumed that their values are also as in (14).

3.1. Bifurcation analysis of the interior equilibrium

The parametric conditions for a locally stable equilibrium in the interior suggest the existence of a Hopf bifurcation, and in this section we prove that numerically. According to Proposition 2 and Remark 1, the interior point P_{xyT} is locally asymptotically stable if and only if $\sigma_2 > \bar{\sigma}_2, \sigma_1 < \bar{\sigma}_1$, and $a_2 a_1 - a_0 > 0$. Liu [45] derived a criterion for the existence of a simple Hopf bifurcation that does not require the eigenvalues of the Jacobian matrix evaluated at the point. Assume that the coordinates of P_{xyT} depend smoothly on a parameter S , for S in an open interval $(0, s)$, and let $\lambda^3 + a_2(S)\lambda^2 + a_1(S)\lambda + a_0(S) = 0$ be the characteristic equation of $J(P_{xyT})$. Also assume that $a_2(S), a_1(S)$ and $a_0(S)$ are smooth in an open interval about $S_H \in (0, s)$. A simple Hopf bifurcation occurs at $S = S_H$ if the following conditions hold

- (a) $a_2(S_H) > 0, a_1(S_H) > 0$ and $\Delta(S_H) = 0$;
- (b) $\left(\frac{d\Delta(S)}{dS}\right)_{S=S_H} \neq 0$.

where $\Delta(S) = a_2(S)a_1(S) - a_0(S)$. It is difficult to write down explicit parametric restrictions ensuring the local asymptotic stability of P_{xyT} but we can discuss it for a specific choice of the model parameters. In the following, we specify a set of parameters and prove the existence of a simple Hopf bifurcation, assuming σ_2 as the bifurcation parameter and checking that the conditions of Liu's criterion are actually verified. Of course, other choices are acceptable for the bifurcation parameter, such as σ_1 or β .

For the parameters (14) we have $\bar{\sigma}_1 = 16/13, \bar{\sigma}_2 = 7/10$ and $\hat{\sigma}_2 = 22/15$. According to Proposition 2, a bifurcation exists for $\sigma_2 > \hat{\sigma}_2$

(note that we have $\sigma_1 < \bar{\sigma}_1$). The coordinates of the interior equilibrium read

$$P_{xyT} = \left(\frac{65\sigma_2 - 80}{10\sigma_2 - 7}, \frac{69}{20\sigma_2 - 14}, 50\sqrt{6}\sqrt{\frac{15\sigma_2 - 22}{10\sigma_2 - 7}} \right)$$

and a_2 and a_1 are positive for $\sigma_2 > 22/15$ (refer to the discussion at the end of Proposition 2 for the expressions of the coefficients of the characteristic polynomial). We also have

$$\Delta(\sigma_2) = \frac{-412425\sigma_2^3 + 1551435\sigma_2^2 - 1859736\sigma_2 + 707712}{16(10\sigma_2 - 7)^3}$$

which is positive in $\left(\frac{22}{15}, \sigma_{2,H}\right)$, negative in $(\sigma_{2,H}, +\infty)$ and vanishes at $\sigma_{2,H} = \frac{5353+23\sqrt{7009}}{4230} \approx 1.285$. It is also $\Delta'(\sigma_{2,H}) \neq 0$, so that the conditions of Liu's criterion are satisfied for a simple Hopf bifurcation to exist at $\sigma_2 = \sigma_{2,H}$. There the interior equilibrium becomes unstable and small-amplitude periodic solutions originate from P_{xyT} .

More generally, from the discussion regarding the stability of P_{xyT} (see Proposition 2), we derive the Hopf curve equation, which reads

$$\left[-\frac{cbe}{2}x_I + T_I^2(\alpha_1\sigma_2e - \alpha_2\sigma_1c)\right]y_I - T_I^2x_I\alpha_1\sigma_1b = 0 \tag{15}$$

where x_I, y_I, T_I stand for the coordinates (6) of the interior equilibrium. When letting a parameter vary, we can solve the equation for a second parameter and obtain a whole bifurcation curve. Fig. 1 shows the curves $\sigma_{2,H}(\sigma_1), \sigma_{2,H}(\beta)$ and $\sigma_{1,H}(\beta)$ (the red lines). It is clear from the picture that horizontal lines at suitable values of the ordinate would intersect the bifurcation curve twice. Correspondingly we can have one or two bifurcations depending on the parameters. As a further numerical example, we consider the solutions of the previous equation in the unknown σ_1 , for $\sigma_2 = 339/200 = 1.695$. Its two solutions are $\sigma_1^{(1)} \approx 0.446$ and $\sigma_1^{(2)} \approx 0.674$. For $\sigma_1 \in [\sigma_1^{(1)}, \sigma_1^{(2)}]$, the corresponding interior equilibria are unstable and accompanied by the formation of limit cycles. When varying σ_1 smoothly in that interval, those cycles produce the surface depicted in Fig. 3. Highlighted in red are the cycles at $\sigma_1 = 0.673, 0.611, 0.496$, while the equilibrium points for $\sigma_1 \in [\sigma_1^{(1)}, \sigma_1^{(2)}]$ lie on the black dashed line. The points H_1 and H_2 are the bifurcation points for $\sigma_1 = \sigma_1^{(1)}$ and $\sigma_1 = \sigma_1^{(2)}$. The figure also shows a trajectory starting slightly apart from the equilibrium and approaching the cycle asymptotically.

Remark 4. The presence of a Hopf bifurcation and resulting limit cycles cause periodic oscillations in population levels, which brings significant challenges for the management of PAs. These fluctuations can lead to ecological instability, as populations may approach critical levels, raising the risk of extinction. Additionally, oscillating populations can impact a tourist's experience, potentially reducing visitor satisfaction if they are unable to observe the species they came to see. This dynamic also complicates management strategies, as the manager must carefully balance revenue generation with conservation goals by adjusting tourist numbers to maintain ecological stability. The variability introduced by limit cycles makes long-term planning difficult and requires continuous monitoring. To mitigate these risks, the manager should adopt an adaptive management approach, closely track ecological and tourism data, and flexibly adjust policies as needed.

3.2. Uniform persistence

In general, persistence means that a species does not go extinct; its population stays above zero over time. The term "uniform" implies that this persistence is consistent across the system and does not depend on specific initial conditions, as long as they are within a certain range (usually positive). Uniform persistence is vital for studying the long-term sustainability and coexistence of species in an ecosystem.

Proposition 3. System (3) is persistent if the equilibrium point P_{xyT} exists and it is not a saddle with two-dimensional manifold

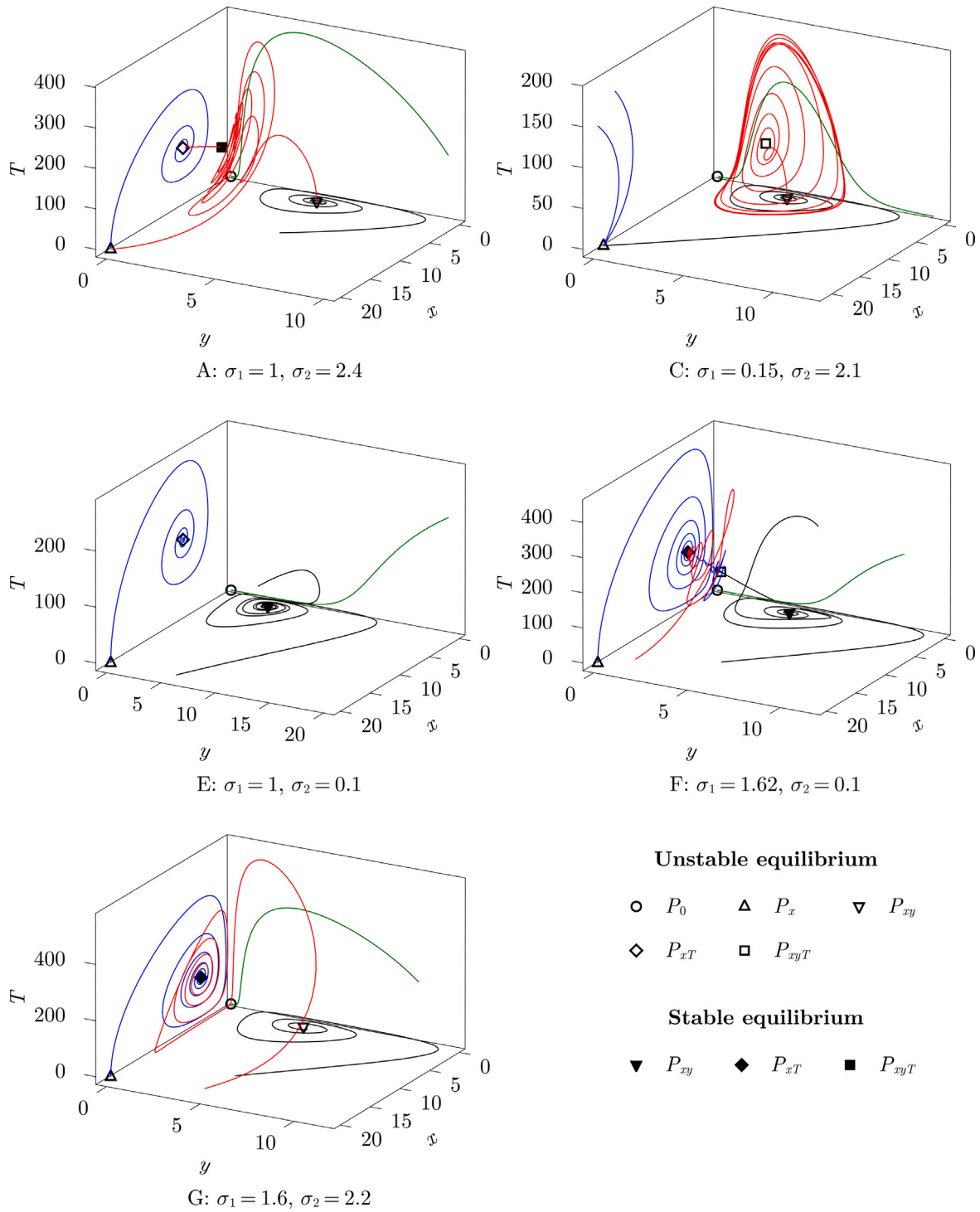


Fig. 2. Sample trajectories in the phase space for choices of the preferences σ_1, σ_2 corresponding to the regions labelled with letters in the top-left panel of Fig. 1.

Proof. For $x \geq 0, y \geq 0, T \geq 0$ consider the positive definite Lyapunov function

$$\psi(x, y, T) = x^{\mu_1} y^{\mu_2} T^{\mu_3}$$

where $\mu_i \geq 0, i = 1, 2, 3$. If we evaluate the logarithmic derivative of $\psi(x, y, T)$ along solution trajectories we obtain

$$\frac{\dot{\psi}}{\psi} = \mu_1(a - bx - cy - \alpha_1 T^2) + \mu_2(-d + ex - \alpha_2 T^2) + \mu_3(-\beta + \sigma_1 x + \sigma_2 y)$$

Remark 2 implies that there are no periodic orbits in the interior of the positive quadrants of the xy -plane and xT -plane. Thus, it is sufficient to demonstrate that for the equilibria P_0, P_x, P_{xy} and P_{xT} , the logarithmic derivatives of ψ are always positive for an appropriate choice of μ_i . The condition is ensured at the origin by the choice $a\mu_1 > d\mu_2 + \beta\mu_3$, while for the remaining equilibrium points we have:

$$P_x : \mu_2(-d + e\frac{a}{b}) + \mu_3(-\beta + \sigma_1\frac{d}{b}) > 0, \quad P_{xy} : \sigma_2 > \hat{\sigma}_2,$$

$$P_{xT} : \sigma_1 < \frac{\beta(ea + \alpha_2)}{\alpha_1 d + a\sigma_2} = \hat{\sigma}_1.$$

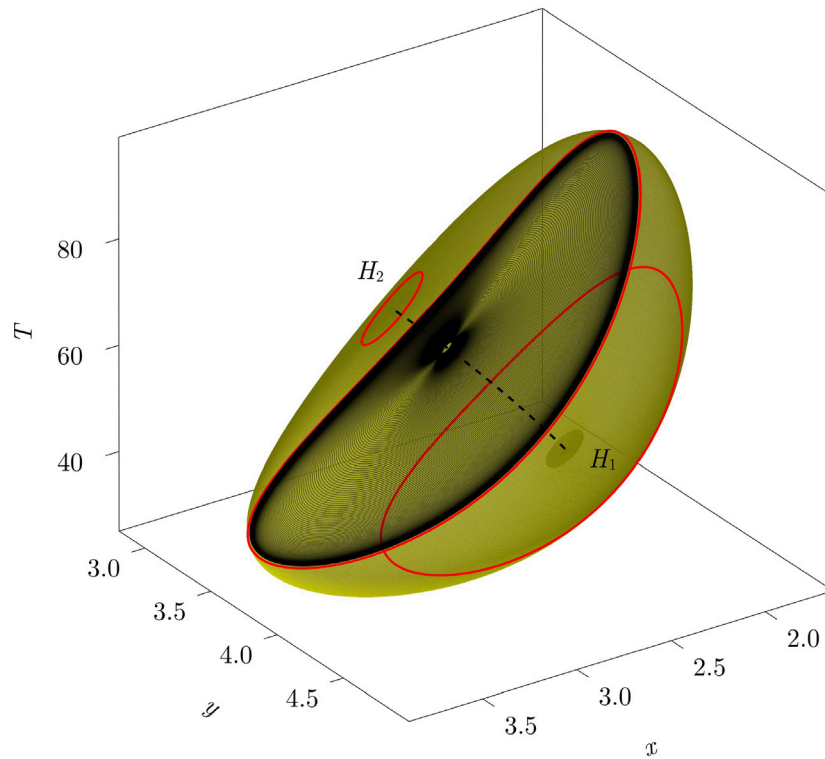


Fig. 3. Surface formed by the limit cycles when varying σ_1 between the values $\sigma_1^{(1)} \approx 0.446$ and $\sigma_1^{(2)} \approx 0.674$ that correspond to the bifurcation points H_1 and H_2 ($\sigma_2 = 339/200$, the other parameters as in (14)). Three sample cycles are highlighted in red. The (unstable) equilibrium points lie on the dashed line. In black, a trajectory originating close to the equilibrium and approaching the limit cycle.

By Remark 1, the above conditions assure that the feasible equilibrium P_{xyT} is not a saddle with two-dimensional stable manifold. ■

By analysing the local asymptotic stability of the equilibrium points at the boundary and taking into account their persistence, we may deduce that all boundary equilibrium points lose stability and just one interior equilibrium point remains.

Furthermore, it is seen that all solutions persist within the first octant, indicating the potential coexistence of species x , y , and tourists T in this scenario. This coexistence may manifest as either a stable equilibrium point or a periodic solution.

4. Sensitivity analysis

4.1. Local sensitivity of the interior equilibrium

Now we turn our attention to the sensitivity of the coordinates of P_{xyT} to the parameters β , σ_1 and σ_2 . The partial derivatives of the expressions (6), or equivalently (9), provide a measure of how much the number of individuals of the two species and of visitors near the interior equilibrium are affected by small changes in the entrance fee and the visitors' preferences. We find the following expressions for the Jacobian matrix of the coordinates of the interior attractor

$$\left(\frac{\partial P_{xyT}}{\partial \beta} \quad \frac{\partial P_{xyT}}{\partial \sigma_1} \quad \frac{\partial P_{xyT}}{\partial \sigma_2} \right) = \frac{1}{(A - C)\sigma_2} \begin{pmatrix} -\frac{\alpha_2}{e} & \frac{\alpha_2}{e}x & \frac{\alpha_2}{e}y \\ A & -Ax & -Ay \\ -\frac{1}{2T} & \frac{x}{2T} & \frac{y}{2T} \end{pmatrix} \quad (16)$$

It is $A > 0$ by definition and $A - C$ must be positive for P_{xyT} to be stable. Keeping into account that the coordinates at the interior equilibrium are positive, we conclude that the derivatives have the signs summarized in Table 2.

In particular, rising β has a negative effect on T , as was clear from the dynamics (3). An increase in the entrance fee produces an increase

Table 2

Variations of the coordinates of P_{xyT} due to variations of the parameters β , σ_1 and σ_2 , as deduced from the expressions (16).

	$\beta \uparrow$	$\sigma_1 \uparrow$	$\sigma_2 \uparrow$
x_I	↓	↑	↑
y_I	↑	↓	↓
T_I	↓	↑	↑

Legend: ↑ increasing; ↓ decreasing.

in the number of predators, this effect becoming smaller in relative terms for larger β , and ultimately a reduction in the number of preys.

Interestingly, increased preferences for both the predators and the preys result in more tourists, greater disturbance to the ecosystem and fewer individuals of both species at first. However, the predators are penalized because, beside being disturbed by tourists, they also find less preys to prey on. At the equilibrium such dynamics goes to the detriment of the predators, which ultimately diminish. The preys ultimately grow, benefiting from the reduced number of predators.

Fig. 4 (left panels) shows the elasticities of the coordinates computed by using the values of the derivatives (16) and of the coordinates at the interior equilibrium, and varying the designated parameter. The right panel of the figure shows the sensitivity of the coordinates at the equilibrium for $\sigma_2 = 1.5$ where we have a stable interior equilibrium. Despite the term “sensitivity” being usually adopted when referring to the derivatives, the values of the bars here represent elasticities, which partially mitigate the difference in magnitude between the derivatives of x_I , y_I and those of T_I . As observed earlier, increasing β obviously decreases the visitors. Less visitors means smaller disturbances to both the preys and the predators, which would increase x and y . But more predators would prey more and, for the given parameters the final balance would be unfavourable to the preys. Increasing the preferences towards the preys or the predators has qualitatively similar effects, but the partial effect of σ_2 is noticeably larger. An increased number of

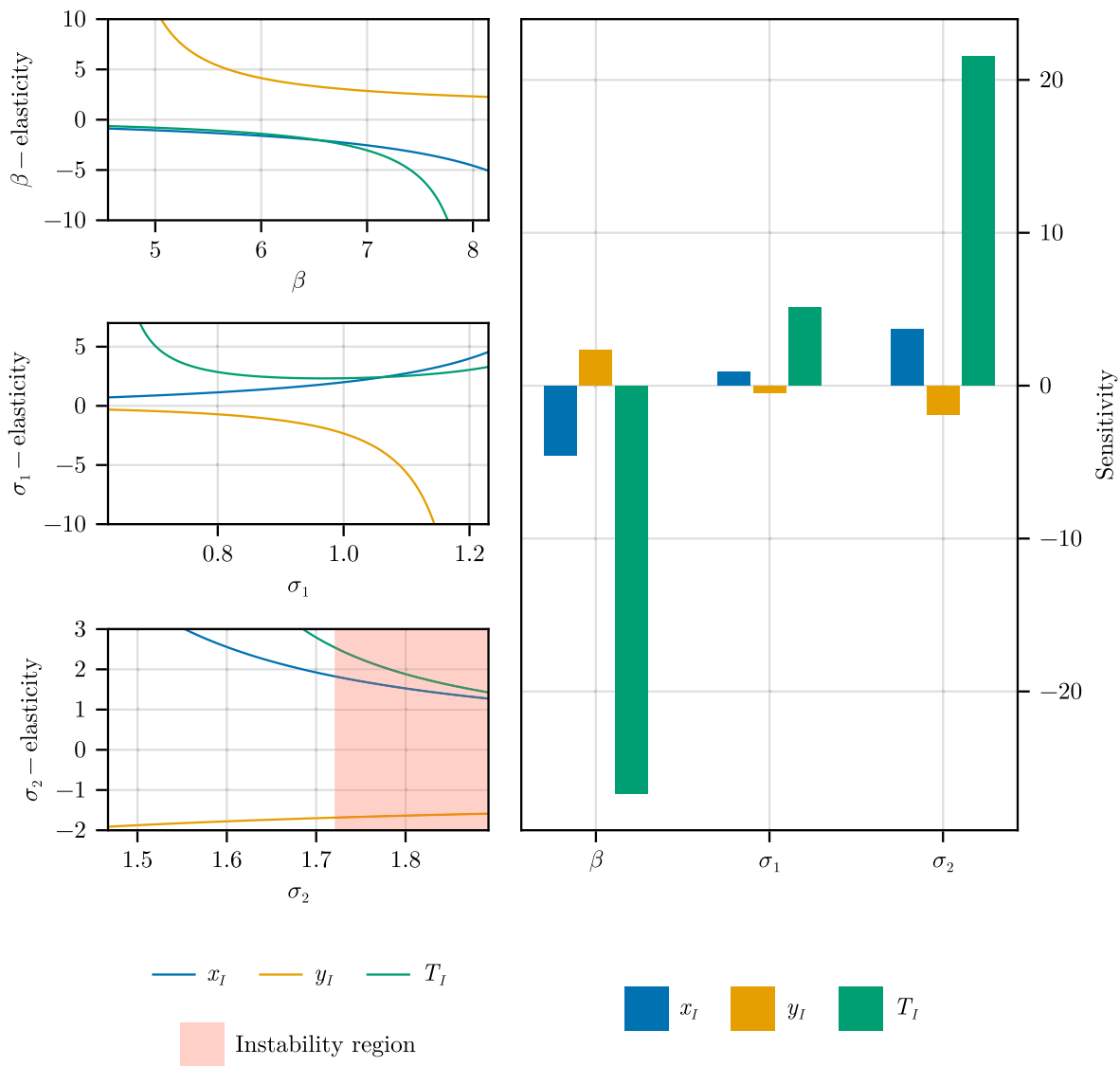


Fig. 4. Elasticities of the coordinates of the interior equilibrium with respect to β , σ_1 , σ_2 . The equilibrium is an attractor outside the shadowed region. $\sigma_2 = 1.5$, the other parameters as in (14).

tourists negatively affects the two species, which would both decrease. But y further decreases when x does, while the reverse is true for x . With the given value of the parameters, the final balance penalizes the predators and favour the preys.

4.2. Global sensitivity analysis

This section focuses on global sensitivity analysis, which is a broader version of local sensitivity analysis. The objective is to comprehensively examine all parameters simultaneously, while also seeing the interplay between the parameters over the whole input space. Through conducting a comprehensive analysis, one may not only see the level of sensitivity of each parameter, but also identify the parameter combinations that have the greatest impact on the output and understand the interactions between the model's parameters.

Out of all the global sensitivity analysis approaches, the Sobol sensitivity analysis based on variance decomposition is now considered one of the most effective.

Sobol's technique relies on decomposing the variance of the model output into individual variances of the input parameters, with increasing dimensionality [46–48]. It assesses the impact of each input parameter and their interactions on the overall variability of the model's output.

Sobol's sensitivity analysis aims to quantify the extent to which the variability in model output is influenced by individual parameters or the interaction between several factors. The breakdown of the output variance in a Sobol sensitivity analysis utilizes the same idea as the conventional analysis of variance in a factorial design.

Let $z = (z_1, z_2, \dots, z_n)$ be independent random input parameters, with joint probability density function

$$P(z_1, z_2, \dots, z_n) = \prod_{i=1}^n p_i(z_i).$$

The model output whose sensitivity to the inputs we want to measure is a function of z , say $f(z)$. If we interpret the parameters in a probabilistic way, then $f(z)$ is random with mean f_0 and variance V

$$f_0 = \int \int \dots \int f(z_1, z_2, \dots, z_n) \prod_{i=1}^n p_i(z_i) dz_i \quad (17)$$

$$V = \int \int \dots \int f^2(z_1, z_2, \dots, z_n) \prod_{i=1}^n p_i(z_i) dz_i - f_0^2 \quad (18)$$

Sobol's technique relies on the decomposition of the variable V into individual contributions from single parameters, combined impacts

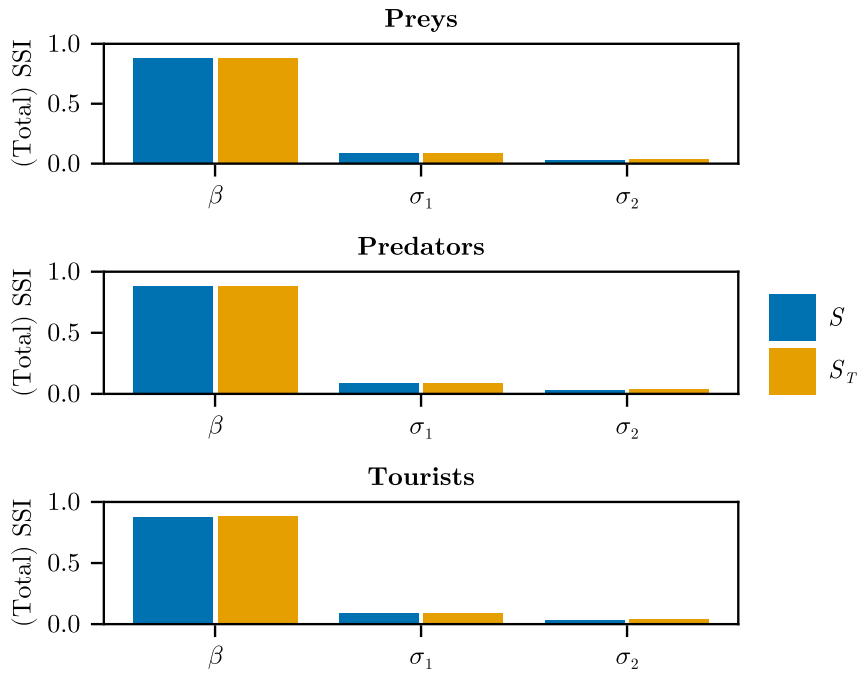


Fig. 5. First-order and total Sobol indices of the interior equilibrium coordinates (same values as Table 3).

Table 3

Sobol indices of the interior equilibrium coordinates x_I, y_I, T_I (9), with model inputs $(\beta, \sigma_1, \sigma_2)$.

	x_I		y_I		T_I	
	S	S_T	S	S_T	S	S_T
β	0.8795913803	0.88349	0.8795902834	0.88318	0.87761268	0.88254145
σ_1	0.08548	0.086652	0.08606	0.08672191	0.08445450	0.08494741
σ_2	0.030771	0.03404912	0.03249013	0.03400991	0.03249013	0.03770102

from pairs of parameters, and so on. Initially, the process involves breaking down the function $f(z)$ into its constituent parts.

$$f(z) = f_0 + \sum_{i=1}^n f_i(z_i) + \sum_{i=1}^n \sum_{j>i}^n f_{ij}(z_i, z_j) + \dots + f_{1,2,\dots,n}(z_1, z_2, \dots, z_n). \quad (19)$$

Let $z = (z_1, z_2, z_3) = (\beta, \sigma_1, \sigma_2)$ be our uniform input variables with $P(\beta, \sigma_1, \sigma_2) = \frac{1}{\beta^M - \beta^m} \cdot \frac{1}{\sigma_1^M - \sigma_1^m} \cdot \frac{1}{\sigma_2^M - \sigma_2^m}$. Let $f(\beta, \sigma_1, \sigma_2)$ be any of

the coordinates x_I, y_I, T_I of the interior equilibrium P_{xyT} (9). Then the output variance decomposes as follows

$$V = V_1 + V_2 + V_3 + V_{12} + V_{13} + V_{23} + V_{123} \quad (20)$$

The *first-order indices* (S) are V_i/V and the *total indices* (S_T) read $S_{i,T} = (V_i + \sum_{j \neq i} V_{ij} + V_{123})/V$, where it is implicit $V_{ij} = V_{ji}$ for $j < i$. We refer the reader to the Appendix for further details and explicit integral expressions for the terms in (19) and (20).

If we consider a symmetric range of $\pm 10\%$ around $\beta = 8$, and of $\pm 5\%$ around $\sigma_1 = 1$ and $\sigma_2 = 2$, we obtain the values in Table 3 for the first-order and total Sobol indices of the interior equilibrium coordinates. Such ranges ensure that the equilibrium is attractive (see the Appendix for the lower and upper bounds of the ranges).

The same values are visualized in Fig. 5. Both the indices are similar for all the coordinates at the equilibrium, which means that the contributions of $\beta\sigma_1, \beta\sigma_2, \sigma_1\sigma_2$ and $\beta\sigma_1\sigma_2$ to the total variance are negligible. In the context of sensitivity analysis using Sobol indices, the values of S_T (88%, 9%, and 3% approximately) suggest the following:

- The fact that the parameter β has a total Sobol index of 88% indicates that it is the dominant factor influencing the output $O_I = (x_I, y_I, T_I)$. This means that 88% of the variability in the output can be attributed to variations in β . Therefore, β has a

significant impact on the model’s behaviour, and controlling or optimizing it will likely lead to the most significant change in the output.

- The parameter σ_1 has a much lower total Sobol index (9%), indicating that it has a smaller but still noticeable effect on the output. While not as dominant as β , changes in σ_1 can still contribute to some degree of variability in the output, but its influence is much weaker compared to the entrance fee.
- With a total Sobol index of 3%, σ_2 has the least influence on the output. This suggests that, even though σ_2 might have some effect, it contributes relatively little to the variability in the model’s behaviour. Therefore, the tourists’ preference for the predator is a less critical parameter when considering how to control or optimize the output.

In summary, if we aim to optimize or control the output “equilibrium point O_I ” focusing on β will likely yield the most significant improvement, whereas σ_1 and σ_2 can be considered secondary factors.

5. Optimization with respect to the entrance fee

5.1. Static optimization

The parameter β can be controlled directly by the managers of the natural park. In principle we may be interested in finding the value β^* that maximizes some objective function $\Pi(x, y, T; \beta)$ for β in the set that guarantees the existence of the stable equilibrium P_{xyT} . The necessary condition for P_{xyT} to exist as an attractor is $\beta \in S_1$, where

$$S_1 = \{\beta > 0 \mid \sigma_1 < \bar{\sigma}_1(\beta) \wedge \sigma_2 > \hat{\sigma}_2(\beta, \sigma_1)\} \\ = \left(\frac{\alpha_1 d + \alpha_2 a}{\alpha_1 e + \alpha_2 b} \cdot \sigma_1, \frac{d}{e} \cdot \sigma_1 + \frac{q_N}{ce} \cdot \sigma_2 \right) = (\bar{\beta}, \beta_{\text{sup}}) \quad (21)$$

The previous set is nonempty because $\sigma_1 < \bar{\sigma}_1 \wedge \sigma_2 > \bar{\sigma}_2$ also imply $\sigma_2 > \bar{\sigma}_2 = \alpha_2 \sigma_1 c / (\alpha_1 e + \alpha_2 b)$ and so $0 < \bar{\beta} < \beta_{sup}$. For P_{xyT} to be stable we also need $\Delta(s) = a_2(s) \cdot a_1(s) - a_0(s) > 0$, as discussed at the end of Proposition 2 and in Section 3.1, where S is a chosen bifurcation parameter in terms of which we can write the coefficients a_2, a_1 and a_0 of the characteristic polynomial for $J(P_{xyT})$. First, observe that for $\beta \in S_1$ we have $x > 0$ and equivalently we can consider the inequality $\tilde{\Delta} = \Delta \cdot e/x > 0$. By making use of the expressions (5), it can be checked directly that $\tilde{\Delta}$ equals the following second-degree polynomial in the variable β

$$\tilde{\Delta} = \frac{p_2}{(\sigma_2(A-C))^2} \beta^2 - \frac{1}{\sigma_2(A-C)} \left[\frac{2p_2}{A-C} \left(y_N + \frac{\sigma_1 d}{\sigma_2 e} \right) + p_1 \right] \beta + \frac{y_N + \frac{\sigma_1 d}{\sigma_2 e}}{A-C} \left[\frac{p_2}{A-C} \left(y_N + \frac{\sigma_1 d}{\sigma_2 e} \right) + p_1 \right] + p_0$$

where

$$p_0 = b d q_N,$$

$$p_1 = q_N \left[\alpha_2 b - \frac{2(\alpha_1 \sigma_2 e - \alpha_2 \sigma_1 c)}{c} \right] + b d (2\alpha_1 \sigma_1 - \alpha_2 b - \alpha_1 e),$$

$$p_2 = \frac{\alpha_2 b + \alpha_1 e}{c} (2\alpha_1 \sigma_2 e - 2\alpha_2 \sigma_1 c - \alpha_2 b c) + 2\alpha_1 \alpha_2 \sigma_1 b$$

The intersection $S_\beta = S_2 \cap S_1$ between the set $S_2 = \{\beta \mid \tilde{\Delta}(\beta) > 0\}$ and the interval (21) provides the region of admissible β for the optimization problem. Because S_1 and S_2 are open sets, there is no guarantee that a maximum of the objective function exists in S_β . In the light of that, we solve the optimization problem in a closed subset $\bar{S}_\beta \subseteq S_\beta$ obtained replacing possible exterior endpoints of S_β with sufficiently close interior points.

As a straightforward application, we consider the quantity $x y T$ as a function of β . In an abstract sense, that may be regarded as a ‘‘fitness’’ function of the park. The product $x y$ may be interpreted as a proxy of biodiversity, which is then multiplied by T . Indeed, in the mission of PAs is the promotion of responsible tourism and in principle we would like to maximize biodiversity and tourism affluence as well. Thus we consider the following problem of maximization of the (log) fitness

$$\max_{\beta \in \bar{S}_\beta} (\log x + \log y + \log T)$$

Replacing the expressions (6) shows that $\Pi(\beta) = \log x + \log y + \log T$ is concave and always has a maximum at β_1 with $\beta_1 \in S_1$. So $\Pi(\beta_1)$ solves the constrained optimization problem as long as $\beta_1 \in S_\beta$, otherwise the constrained maximum is reached at one of the endpoints of \bar{S}_β . In Fig. 6 we show the values of β^* and the maximum of Π when varying the preferences σ_1 and σ_2 . Within the red lines are points for which $\beta_1 \in S_\beta$ and then $\beta^* = \beta_1$. Elsewhere it is $\beta_1 \in S_1$ but $\beta_1 \notin S_2$, that is β_1 corresponds to an unstable equilibrium. So β^* equals one of the endpoints of \bar{S}_2 and the maximum is attained at the boundaries of the stability region of P_{xyT} . Interestingly, the optimal fitness $\Pi(\beta^*)$ exhibits small variations across σ_1 and σ_2 (right panel). At the same time optimality requires slightly large variations of β (left panel). Remarkably, the preference for the preys has a larger impact on β^* , even though a higher σ_1 comes with a higher σ_2 if we restrict to the region where a maximum exists in the interior of S_β .

5.2. Optimal control

As defined in the Introduction, the two main objectives in managing a PA are to preserve the species living within it and, at the same time, to make the area accessible to visitors. In this regard, the park manager is interested in optimizing a certain utility function U , using the entrance fee $\beta \in [\beta_{min}, \beta_{max}]$ as the control variable. The control problem over an infinite time horizon is given by

$$\max_{\beta \in [\beta_{min}, \beta_{max}]} J(\beta) = \max_{\beta \in [\beta_{min}, \beta_{max}]} \int_0^{+\infty} U(t, x, y, T, \beta) e^{-\delta t} dt \quad (22)$$

subject to the constraints (3) and $x_0 > 0, y_0 > 0, T_0 > 0$.

The function U can be thought of as a logarithmic transformation of the Cobb–Douglas utility function, which has the characteristics of concavity and monotonicity. So $U(t, x, y, T, \beta) = p \log x + q \log y + r \log T$, where p, q , and r are strictly positive parameters, see [49]. With this choice, strictly positive values of x, y, T are required. Indeed the partial derivatives of U with respect to x, y and T go to $+\infty$ if x, y or T go to zero respectively (*ceteris paribus*), and the agent would suffer an infinite utility loss. Moreover, the agent cares for biodiversity [49]. The parameter δ indicates the subjective discount rate ($\delta > 0$).³ The associated Hamiltonian function is given by

$$\mathcal{H}(t, x, y, T, \beta) = (p \log x + q \log y + r \log T) e^{-\delta t} + \lambda_1 x(a - bx - cy - \alpha_1 T^2) + \lambda_2 y(-d + ex - \alpha_2 T^2) + \lambda_3 T(-\beta + \sigma_1 x + \sigma_2 y)$$

where $\lambda_1, \lambda_2, \lambda_3$ are adjoint variables corresponding to the states x, y, T respectively. The Hamiltonian is linear in the control variable β and then the optimal strategy involves singular and bang–bang controls. The control $\beta(t)$ that maximizes \mathcal{H} (optimal control) must satisfy the following condition:

$$\beta^o(t) = \begin{cases} \bar{\beta}, & \text{if } \frac{\partial \mathcal{H}}{\partial \beta} \neq 0 \\ \beta^*, & \text{if } \frac{\partial \mathcal{H}}{\partial \beta} = 0 \end{cases} \quad (23)$$

The optimal control (23) suggests dividing the problem into two sub-problems [50,51]. The first involves finding the minimum time (τ) necessary for the system (i.e., the trajectories $X(t) = (x(t), y(t), T(t))$) to reach the internal equilibrium, $X_I = (x_I, y_I, T_I)$, through the bang–bang control policy ($\bar{\beta}$), which involves alternating between the entrance fees β_{min} and β_{max} . The second sub-problem, once equilibrium is reached, i.e., $X(\tau) = X_I$, is to apply the singular control $\beta(t) = \beta^*$ for $t > \tau$.

In order to solve the second sub-problem (referred to in [52,53]), we applied Pontryagin’s Maximum Principle, where the system

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial \mathcal{H}}{\partial x} = -T \sigma_1 \lambda_3 - \lambda_1 (-T^2 \alpha_1 - bx - cy + a) + \lambda_1 x b - \lambda_2 y e - \frac{p e^{-\delta t}}{x} \\ \dot{\lambda}_2 &= -\frac{\partial \mathcal{H}}{\partial y} = -T \sigma_2 \lambda_3 + \lambda_1 x c - \lambda_2 (-T^2 \alpha_2 + ex - d) - \frac{q e^{-\delta t}}{y} \\ \dot{\lambda}_3 &= -\frac{\partial \mathcal{H}}{\partial T} = -(\sigma_1 x + \sigma_2 y - \beta) \lambda_3 + 2 \lambda_1 x T \alpha_1 + 2 \lambda_2 y T \alpha_2 - \frac{r e^{-\delta t}}{T}. \end{aligned} \quad (24)$$

represents the dynamics of adjoint variables and $\frac{\partial \mathcal{H}}{\partial \beta} = 0$ gives

$$-T \lambda_3 = 0 \Rightarrow \lambda_3(t) = 0.$$

The singular control must also satisfy equations (once the system has reached the equilibrium point, so $\frac{\partial \mathcal{H}}{\partial \beta} = 0$)

$$\begin{cases} a - bx - cy - \alpha_1 T^2 = 0 \\ -d + ex - \alpha_2 T^2 = 0 \\ -\beta + \sigma_1 x + \sigma_2 y = 0. \end{cases} \quad (25)$$

From the Eqs. (24) along with steady state (25) gives

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_1 x_I b - \lambda_2 y_I e - \frac{p e^{-\delta t}}{x_I} \\ \dot{\lambda}_2 &= \lambda_1 x_I c - \frac{q e^{-\delta t}}{y_I} \\ \dot{\lambda}_3 &= 2 \lambda_1 x_I T_I \alpha_1 + 2 \lambda_2 y_I T_I \alpha_2 - \frac{r e^{-\delta t}}{T_I} = 0 \end{aligned} \quad (26)$$

³ The discount rate (δ) applies a weighting factor to future payoffs, reflecting the idea that a payoff received sooner is more valuable than one received later. The discount rate is key in determining the present value of future utility cumulated over an infinite time horizon. If δ is high, future utility is discounted more heavily, making near-term benefits more influential in determining the optimal policy. If δ is low, future outcomes have relatively more weight, emphasizing the importance of long-term outcomes in decision-making. Thus, in an infinite-horizon control problem, the discount rate’s function is twofold: it serves both as a mechanism to manage the trade-off between present and future payoffs and as a mathematical tool to ensure convergence of the objective function.

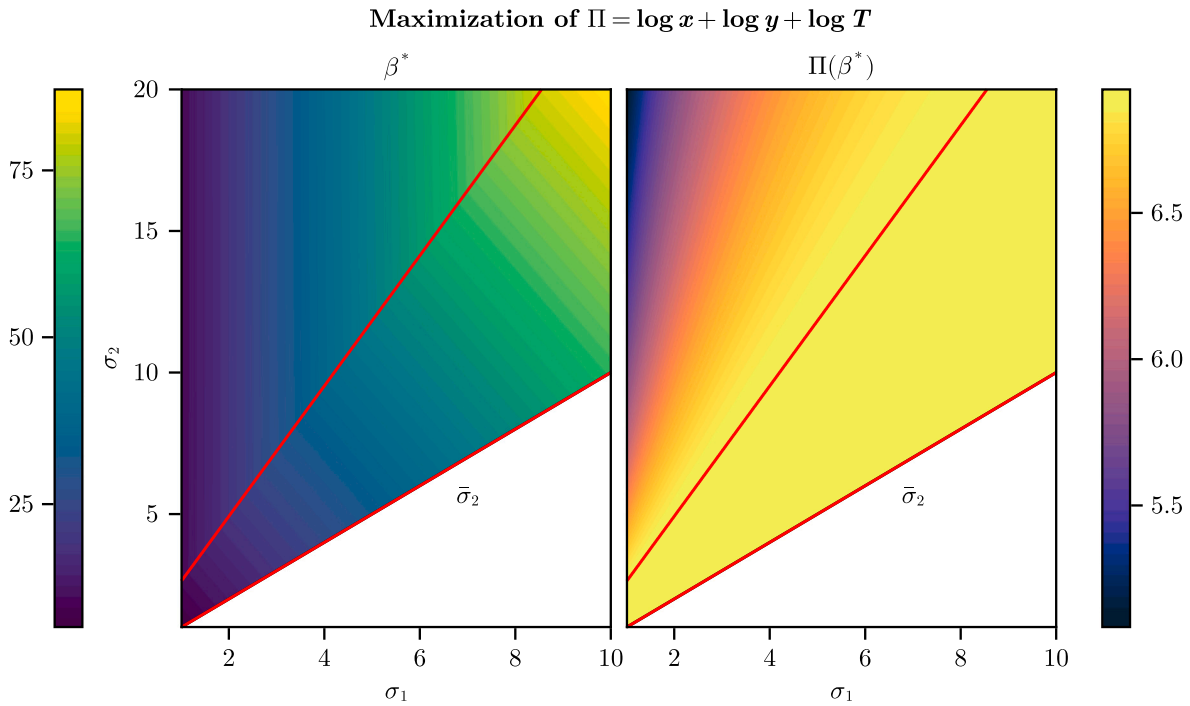


Fig. 6. Static optimization of the objective function $\log x + \log y + \log T$ at the interior attractor.

where we have put $\lambda_3(t) = 0$. The system of first-order linear ordinary differential equations defined by the first and second equation in (26) can now be solved by applying the operator method. Elimination of λ_1 leads to the equation

$$[D^2 - bx_I D + cex_I y_I] \lambda_2 = P_1 e^{-\delta t} \tag{27}$$

where $P_1 = (\delta - bx_I) \frac{q}{y_I} - pc$ and $D = \frac{d}{dt}$. The complete solution of Eq. (27) is:

$$\lambda_2(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t} + \frac{P_1}{Q} e^{-\delta t} \tag{28}$$

where C_1, C_2 are arbitrary constants, and $Q = (\delta^2 + bx_I \delta + cex_I y_I) \neq 0$
 $m_j = \frac{1}{2} (bx_I \pm \sqrt{(bx_I)^2 - 4cex_I y_I})$, $j = 1, 2$.

We note that when $t \rightarrow +\infty$, then the shadow price $\lambda_2(t)$ is bounded if $C_1 = C_2 = 0$, so

$$\lambda_2(t) = \frac{P_1}{Q} e^{-\delta t}.$$

Proceeding in a similar manner, we have:

$$\lambda_1(t) = \frac{P_2}{Q} e^{-\delta t}$$

where $P_2 = eq + \frac{\delta p}{x_I}$.

Replacing $\lambda_1(t), \lambda_2(t)$ in the third equation of (26), we get

$$-\delta^2 r + ((2\alpha_1 p + 2\alpha_2 q)(T_I)^2 - brx_I) \delta + (2q(\alpha_1 e + \alpha_2 b)x_I - 2\alpha_2 cpy_I)(T_I)^2 - cex_I y_I = 0 \tag{29}$$

The above equation gives the desired singular path (see [50,51]). Using the expressions of the coordinates x_I, y_I, T_I of the interior equilibrium point into (29) we obtain the equation for β .

The optimal fee policy is given by

$$\beta^0(t) = \begin{cases} \tilde{\beta}(t), & t \in [0, \tau], \\ \beta^*, & t > \tau, \end{cases} \tag{30}$$

with the optimal path

$$\Gamma^0(t) = \begin{cases} \tilde{\Gamma}(t), & t \in [0, \tau] \\ X_I, & t > \tau, \end{cases} \tag{31}$$

where $\tilde{\Gamma}(t)$ is the solution of the minimum-time control problem

$$\min_{\beta \in [\beta_{min}, \beta_{max}]} \int^t f$$

subject to the constraints (3) and $x_0 > 0, y_0 > 0, T_0 > 0$.

Numerical example. We set the parameter values⁴ as follows: $a = 5, b = 0.25, c = 1, d = 1, e = 0.5, p = q = r = 1, \alpha_1 = \frac{3}{20000}, \alpha_2 = \frac{1}{10000}, \sigma_1 = 1, \sigma_2 = 2, \delta = \frac{1}{100}$. Then Eq. (29) becomes

$$\frac{1}{4} \beta^2 - \frac{20509}{400} \beta + \frac{642481}{2500} = 0 \tag{32}$$

which has the roots $\beta_1 = 8.718983719, \beta_2 = 11.79001628$. For our choice of the parameters, the admissible values for β are those in $S_\beta = (6.5, 11)$, see Section 5.1. So β_1 is the only feasible value as $\beta_2 \notin S_\beta$.

Hence $\beta^* = \beta_1$ and the equilibrium point $(x_I, y_I, T_I) = (4.281016281, 2.218983719, 106.7945757)$ is reached at time $\tau = 0.4591$ for the initial conditions $x_0 = 0.7575, y_0 = 0.7234, T_0 = 113.9508$. Thus, the optimal fee policy results

$$\beta^0(t) = \begin{cases} 6.5, & \text{when } t \in [0, 0.2268], \\ 11, & \text{when } t \in [0.2268, 0.4591], \\ \beta^*, & \text{when } t > \tau. \end{cases} \tag{33}$$

where we chose β_{min} and β_{max} close to the endpoints of the admissible range S_β .

Fig. 7 shows the results. In the top-left panel the evolution of the controlled variables is represented. The number of prey (blue line) has the maximum for $t = 0.1$, while the number of predator (orange line) and tourists (green line) have the maximum for $t = 0.23$, corresponding to the β_{min} fee (6.5). The number of predators and tourists is decreasing for $0.23 \leq t \leq 0.46$, corresponding to the β_{max} fee (8.72). In the top right panel the fee policy (33) is represented. The bottom panel shows the trajectory of the uncontrolled system (3) converging to the stable equilibrium P_{xyT} (blue line). This point is the attractor that corresponds to $\beta = \beta^*$ and is reached only for $t \rightarrow +\infty$. In the same panel,

⁴ In particular, we chose $p = q = r = 1$ in order to maximize the same fitness function $\log x + \log y + \log T$, already considered in a static optimization setting in Section 5.1.

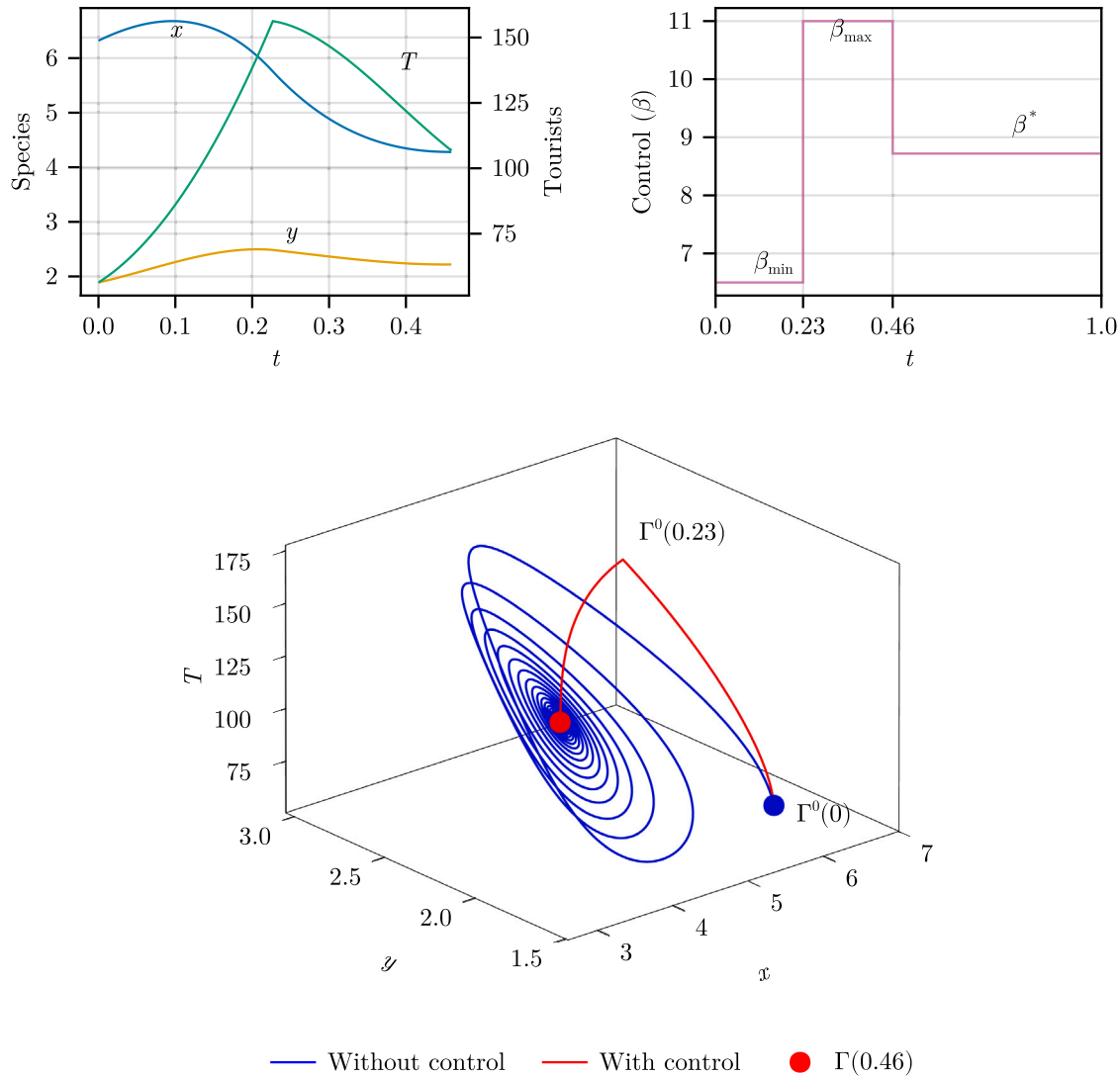


Fig. 7. Top panels: time evolution of the controlled variables (left) and values of the control applied (right). Bottom: trajectory of the uncontrolled system (3) converging to the P_{xyT} equilibrium (blue line); controlled system (31) converging to the same equilibrium (red line). $\beta = 8.719$, $\sigma_1 = 1$, $\sigma_2 = 2$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the trajectory Γ^0 of the controlled system is also shown (red line). It is important to stress that Γ^0 reaches the same equilibrium P_{xyT} of the uncontrolled system in finite time, exactly for $t = \tau$, and then it stays there indefinitely. In particular, the optimal fee policy allows the system to reach the attractive equilibrium in minimum time, not just to approach it asymptotically. The trajectory $\bar{\Gamma}(t)$ from $t = 0$ to $t = \tau$ (bottom panel), corresponding to the optimal fee policy, is by construction the trajectory over which the cumulated discounted utility in (22) attains its maximum. From a practical viewpoint, this control exercise provides a mean to drive the system towards the equilibrium state in a way that guarantees the maximum discounted utility. Also, τ can be regarded as estimating the minimum time for the system to reach P_{xyT} and β as a control parameter instrumental in computing such an estimate.

6. Conclusions

In recent years, the literature on PAs has shifted towards understanding their dual role of preserving natural environments and promoting recreational and cultural activities. The emergence of ecotourism has become a significant source of funding for PAs, with a positive impact on both nature conservation and community development. Revenues generated from tourism can be utilized to preserve natural environments and biodiversity.

In this study, we introduce a three-dimensional Lotka–Volterra model that not only considers the dynamics of prey and predator populations, but also incorporates tourists. The dynamics of tourists is influenced by the cost of the entrance fee and by the preferences for observing the prey and predator species. The equilibrium points of the system are analysed through conditions of existence and stability. The parametric conditions for a locally stable equilibrium in the interior indicate the existence of a Hopf bifurcation, which is proved numerically.

A sensitivity analysis is performed at the interior equilibrium with respect to the entrance fee (β) and the preferences for the two species

(σ_1 and σ_2). The entrance fee can be controlled directly by the park manager and its increase has a negative impact on the number of tourists. But that also leads to an increase in the number of predators and to a reduction in the number of preys at last. On the other hand, higher preferences for both species lead to more tourism. This comes to the detriment of the predators, which are deemed to diminish. The stability analysis of the dynamical system also suggests that a strong preference for the preys may lead to the extinction of the predators. At the same time, the preference for the predators is crucial in maintaining a positive number of tourists. In terms of the fee parameter, for an attractive equilibrium to exist we find that β is bound by the values of the preferences. Outside such range, the extinction of the predators or of the tourists can be expected.

A static sensitivity analysis confirms that rising β has a negative effect on T and ultimately determines a reduction of the number of preys. Increased preferences determine more tourists and disturbance for the species but, at the equilibrium, the dynamics favour the preys which ultimately grow while the predators diminish. Then a global sensitivity analysis is performed by variance decomposition using Sobol's method. The entrance fee β turns out to be the most influential model input, determining 88% of the model output, with σ_1 and σ_2 accounting for just 9% and 3% respectively.

A static optimization at the interior equilibrium is performed to maximize the "fitness" of the park in terms of the numbers of preys, predators and tourists as well, subject to the constraint that the equilibrium remains an attractor. This shows that the maximum of the fitness is rather stable across values of σ_1 and σ_2 and that the optimal β is mostly influenced by the preference for the preys.

Finally an optimal control policy is implemented that allows the administrators of the park to maximize the discounted cumulated utility using β as a control instrument. By applying this control the park manager ensures that the system reaches in minimum time the same stable equilibrium as in the absence of control, and the corresponding system's trajectory maximizes the total discounted benefit at the same time. As a by-product, the optimal fee policy provides a way to indirectly estimate a lower bound on the time required for the system to reach the equilibrium.

We believe our study contributes to clarify the interplay between tourist preferences and entrance fee in PAs. The theoretical analysis is also able to provide relevant policy indications as far as the problem of tuning β in relation to the preferences is concerned. More generally it provides policy makers a powerful framework to analyse the possible dynamical scenarios that arise depending on the tourists' preferences for the species, for every choice of β . This is accomplished through the analysis of a proposed three-dimensional dynamical model that provides a reasonable, albeit simplified, picture of the interactions between the tourists and the species.

As a future perspective, comparison with data-driven models and methodologies may provide new insights and suggest further terms of interactions to include in the model's equations (even if at the cost of losing complete analytical tractability). Also, attempts could be made to fit the model against real-world (x, y, T) time-series collected at specific PAs, possibly using exogenous estimates to fix some of the parameters (e.g. ecological or biological studies considering the natural dynamics without tourists, or survey studies estimating tourists' preferences).

CRedit authorship contribution statement

Danilo Delpini: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Formal analysis. **Roberta Melis:** Writing – review & editing, Writing – original draft, Validation, Resources, Formal analysis. **Paolo Russu:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Roberta Melis reports financial support was provided by Fondazione di Sardegna. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The research was funded by Fondazione di Sardegna, funding 2018–2020 and 2021, under the project "Modeling and Simulation of Dynamical Processes and Agent Behavior on Networks". Unique project identifier (CUP) J85F21002630007.

Appendix

Formula (19) is called ANOVA (Analysis Of Variances)-representation of $f(z)$ if

$$\int_{\mathcal{P}} f_{i_1, i_2, \dots, i_n}(z_1, z_2, \dots, z_n) \prod_{k=1}^n p_k(z_k) dz_k = 0, \quad \text{for } k = i_1, i_2, \dots, i_n. \quad (34)$$

It follows from (34) that the members in (19) are orthogonal and can be expressed as integrals of $f(z)$.

Let us set the parameter values as : $a = 5, b = 0.25, c = 1, d = 1, e = 0.5, \alpha_1 = 0.00015, \alpha_2 = 0.0001$, then the first equation in (9) became $x_I = \frac{2\beta - 13\sigma_2}{2\sigma_1 - 2\sigma_2}$.

Then the summands in (19), with $f(z) = x_I$, can be computed as:

$$\begin{aligned} f_0 &= \frac{1}{(\beta^M - \beta^m)(\sigma_1^M - \sigma_1^m)(\sigma_2^M - \sigma_2^m)} \\ &\quad \times \int_{\beta^m}^{\beta^M} \int_{\sigma_1^m}^{\sigma_1^M} \int_{\sigma_2^m}^{\sigma_2^M} x_I d\beta d\sigma_1 d\sigma_2 \\ f_1(\beta) &= \frac{1}{(\sigma_1^M - \sigma_1^m)(\sigma_2^M - \sigma_2^m)} \int_{\sigma_1^m}^{\sigma_1^M} \int_{\sigma_2^m}^{\sigma_2^M} x_I d\sigma_1 d\sigma_2 - f_0 \\ f_2(\sigma_1) &= \frac{1}{(\beta^M - \beta^m)(\sigma_2^M - \sigma_2^m)} \int_{\beta^m}^{\beta^M} \int_{\sigma_2^m}^{\sigma_2^M} x_I d\beta d\sigma_2 - f_0 \\ f_3(\sigma_2) &= \frac{1}{(\beta^M - \beta^m)(\sigma_1^M - \sigma_1^m)} \int_{\beta^m}^{\beta^M} \int_{\sigma_1^m}^{\sigma_1^M} x_I d\beta d\sigma_1 - f_0 \\ E_1(\sigma_1, \sigma_2) &= \frac{\int_{\beta^m}^{\beta^M} x_I d\beta}{\beta^M - \beta^m}, \quad E_2(\beta, \sigma_2) = \frac{\int_{\sigma_1^m}^{\sigma_1^M} x_I d\sigma_1}{\sigma_1^M - \sigma_1^m}, \\ E_3(\beta, \sigma_1) &= \frac{\int_{\sigma_2^m}^{\sigma_2^M} x_I d\sigma_2}{\sigma_2^M - \sigma_2^m} \\ f_{1,2} &= E_3 - f_1 - f_2 - f_0, \quad f_{1,3} = E_2 - f_1 - f_3 - f_0, \\ f_{2,3} &= E_1 - f_2 - f_3 - f_0 \\ f_{1,2,3} &= x_I - f_1 - f_2 - f_3 - f_{1,2} - f_{1,3} - f_{2,3} - f_0 \end{aligned}$$

The variances can be computed as:

$$\begin{aligned} V_1 &= \frac{\int_{\beta^m}^{\beta^M} f_1^2 d\beta}{\beta^M - \beta^m}, \quad V_2 = \frac{\int_{\sigma_1^m}^{\sigma_1^M} f_2^2 d\sigma_1}{\sigma_1^M - \sigma_1^m}, \quad V_3 = \frac{\int_{\sigma_2^m}^{\sigma_2^M} f_3^2 d\sigma_2}{\sigma_2^M - \sigma_2^m} \\ V_{1,2} &= \frac{1}{(\beta^M - \beta^m)(\sigma_1^M - \sigma_1^m)} \int_{\beta^m}^{\beta^M} \int_{\sigma_1^m}^{\sigma_1^M} f_{1,2}^2 d\beta d\sigma_1 \\ V_{1,3} &= \frac{1}{(\beta^M - \beta^m)(\sigma_1^M - \sigma_1^m)} \int_{\beta^m}^{\beta^M} \int_{\sigma_2^m}^{\sigma_2^M} f_{1,3}^2 d\beta d\sigma_2 \\ V_{2,3} &= \frac{1}{(\sigma_1^M - \sigma_1^m)(\sigma_2^M - \sigma_2^m)} \int_{\sigma_1^m}^{\sigma_1^M} \int_{\sigma_2^m}^{\sigma_2^M} f_{2,3}^2 d\sigma_1 d\sigma_2 \end{aligned}$$

$$V_{1,2,3} = \frac{1}{(\beta^M - \beta^m)(\sigma_1^M - \sigma_1^m)(\sigma_2^M - \sigma_2^m)} \int_{\beta^m}^{\beta^M} \int_{\sigma_1^m}^{\sigma_1^M} \int_{\sigma_2^m}^{\sigma_2^M} f_{1,2,3}^2 d\beta d\sigma_1 d\sigma_2$$

$$V = V_1 + V_2 + V_3 + V_{1,2} + V_{1,3} + V_{2,3} + V_{1,2,3}.$$

Thus, the *first-order* (S) Sobol indices are:

$$S_1 = \frac{V_1}{V}, \quad S_2 = \frac{V_2}{V}, \quad S_3 = \frac{V_3}{V}$$

while, the *total* (S_T) Sobol indices are:

$$S_{1T} = \frac{V_1 + V_{1,2} + V_{1,3} + V_{1,2,3}}{V}, \quad S_{2T} = \frac{V_2 + V_{1,2} + V_{2,3} + V_{1,2,3}}{V},$$

$$S_{3T} = \frac{V_3 + V_{1,3} + V_{2,3} + V_{1,2,3}}{V}.$$

Replacing, $\beta^m = \frac{36}{5}$, $\beta^M = \frac{44}{5}$, $\sigma_1^m = \frac{19}{20}$, $\sigma_1^M = \frac{21}{20}$, $\sigma_2^m = \frac{19}{10}$, $\sigma_2^M = \frac{21}{10}$,

we obtain the first two columns of Table 3. We proceed in a similar way for $f(z) = y_I = \frac{-2\beta+13\sigma_1}{2\sigma_1-2\sigma_2}$ and $f(z) = T_I = 50\sqrt{\frac{2\beta-4\sigma_1-9\sigma_2}{\sigma_1-\sigma_2}}$.

Remark 5. The integrals involving the model outputs x_I and y_I were calculated analytically, while for T_I they were estimated numerically using Jansen's formulas [54].

Data availability

No data was used for the research described in the article.

References

- [1] Dudley N. Guidelines for applying protected area management categories. Vol. 21, Gland, Switzerland: IUCN; 2008.
- [2] Liu T, Tien C. Assessing tourists' preferences of negative externalities of environmental management programs: A case study on invasive species in shei-pa national park, taiwan. Sustainability 2019;11(2953):155–65.
- [3] Eagles PFJ, McCool SF, Haynes CD. Sustainable tourism in protected areas: guidelines for planning and management. IUCN – The World Conservation Union; 2002.
- [4] Buckley R. Pay to play in parks: An australian policy perspective on visitor fees in public protected areas. J Sustain Tour 2003;11(1):56–73.
- [5] Drumm A. Threshold of sustainability for protected areas. BioScience 2008;58(9):782–3.
- [6] Yoon HV, Zou SS. An empirical investigation of the effects of entrance fees on national park visitors. Tour Recreat Res 2023;1–11.
- [7] Lupi F, von Haefen RH, Cheng L. Distributional effects of entry fees and taxation for financing public beaches. Land Econom 2022;98(3):509–19.
- [8] Gao Q, Cui S, Shi P, et al. Exploring tourists' preferences and willingness to pay for national park recreation improvements based on regret and utility comparison. Sci Rep 2024;14:21524.
- [9] Sarkar B, Bhattacharya S, Bairagi N. An ecological-economic fishery model: Maximizing the societal benefit through an integrated approach of fishing and ecotourism. Math Methods Appl Sci 2023;46:14962–82.
- [10] Notaro S, Grilli G. Assessing tourists' preferences for conservation of large carnivores in the italian alps using a discrete choice experiment. J Environ Plan Manag 2022;65(7):1261–80.
- [11] Russu P. Hopf bifurcation in an environmental defensive expenditures model with time delay. Chaos Solitons Fractals 2009;42(5):3147–59.
- [12] Kerley G, Geach B, Vial C. Jumbos or bust: Do tourists' perceptions lead to an under-appreciation of biodiversity? S Afr J Wildl Res 2003;33(1):13–21.
- [13] Lindsey P, Alexander R, Mills M, nach SR, Woodrooffe R. Wildlife viewing preferences of visitors to protected areas in south africa: implications for the role of ecotourism in conservation. J Ecotour 2009;6(1):19–33.
- [14] Minin ED, Fraser I, Slotow R, MacMillan D. Understanding heterogeneous preference of tourists for big game species: implications for conservation and management. Animal Conserv 2013;249–58.
- [15] Maciejewski K, Kerley G. Understanding tourists' preference for mammal species in private protected areas: Is there a case for extralimital species for ecotourism? PLoS ONE 2014;9(2):e88192. <http://dx.doi.org/10.1371/journal.pone.0088192>.
- [16] Guo Y, Fennell D. Preference for animals: A comparison of first-time and repeat visitors. J Zool Bot Gard 2024;5(1):19–35.
- [17] Obradović S, Stojanović V, Tešin A, Šećerov I, Pantelić M, Dolinaj D. Memorable tourist experiences in national parks: Impacts on future intentions and environmentally responsible behavior. Sustainability 2023;15(1).
- [18] Wallace R. Conservation or preservation? Protected areas, sustainability, and the challenges of framing. In: Weder F, Krainer L, Karmasin M, editors. The sustainability communication reader. Wiesbaden: Springer VS; 2021, p. 373–93.
- [19] Mzek T, Samdin Z, Mohamad WNW. Assessing visitors' preferences and willingness to pay for the malayan tiger conservation in a malaysian national park: A choice experiment method. Ecol Econom 2022;191:107218.
- [20] Eyster HN, Naidoo R, Chan KMA. Not just the big five: African ecotourists prefer parks brimming with bird diversity. Animal Conserv 2023;26(4):410–9.
- [21] Altan A, Karasu S, Bekiros S. Digital currency forecasting with chaotic meta-heuristic bio-inspired signal processing techniques. Chaos Solitons Fractals 2019;126:325–36.
- [22] Pekkaya M, Uysal Z, Altan A, Karasu S. Artificial intelligence-based evaluation of the factors affecting the sales of an iron and steel company. Turk J Electr Eng Comput Sci 2024;32(1):51–67.
- [23] Lotka AJ. Elements of physical biology. Sci Prog Twentieth Century (1919-1933) 1926;21(82):341–3.
- [24] Volterra V. Fluctuations in the abundance of a species considered mathematically. Nature 1926;118(2972):558–60.
- [25] Xin S, Li L, Nie H. The effect of advection on a predator–prey model in open advective environments. Commun Nonlinear Sci Numer Simul 2022;113:106567.
- [26] Sarif N, Kumar A, Anshu, Sarwadi S, Dubey B. Spatio-temporal dynamics in a delayed prey–predator model with nonlinear prey refuge and harvesting. Chaos Solitons Fractals 2024;186:115247.
- [27] Chu C, Liu W, Lv G, Moussaoui A, Auger P. Optimal harvest for predator–prey fishery models with variable price and marine protected area. Commun Nonlinear Sci Numer Simul 2024;134:107992.
- [28] Ahmed N, Yasin MW, Baleanu D, Tintareanu-Mircea O, Iqbal MS, Akgül A. Pattern formation and analysis of reaction–diffusion ratio-dependent prey–predator model with harvesting in predator. Chaos Solitons Fractals 2024;186:115164.
- [29] Du Y, Sui M. Stability and spatiotemporal dynamics in a diffusive predator–prey model with nonlocal prey competition and nonlocal fear effect. Chaos Solitons Fractals 2024;188:115497.
- [30] Bazykin AD. Nonlinear dynamics of interacting populations. Vol. 11, World Scientific; 1998.
- [31] Guan XY, Liu Y, Xie DX. Stability analysis of a lotka-volterra type predator–prey system with allele effect on the predator species. Commun Math Biol Neurosci 2018;2018:9.
- [32] Bakare EA, Chakraverty S, Abolarin OE. Qualitative analysis and homotopy based solution of two species lotka-volterra model. Int J Pure Appl Math 2018;119(2):261–80.
- [33] Freedman HI, Waltman P. Mathematical analysis of some three-species food-chain models. Math Biosci 1977;33(3–4):257–76.
- [34] Verhulst P-F. Notice sur la loi que la population suit dans son accroissement. Corresp Math Phys 1838;10:113–21.
- [35] Holling CS. The components of predation as revealed by a study of small-mammal predation of the european pine sawfly. Can Entomol 1959;91(5):293–320.
- [36] Liu J, Wu Y, Jiang X, Jin D. Tourists' preferences and willingness to pay for biodiversity, concession activity and recreational management in wuyishan national park in china: A choice experiment method. Forests 2024;15(4):629.
- [37] Tucker MA, et al. Moving in the anthropocene: Global reductions in terrestrial mammalian movements. Science 2018;359:466–9.
- [38] Gaynor KM, Hojnowski CE, Carter NH, Brashares JS. The influence of human disturbance on wildlife nocturnality. Science 2018;360:1232–5.
- [39] Smith JA, Suraci JP, Clinchy M, Crawford A, Roberts D, Zanette LY, Wilmers CC. Fear of the human 'super predator' reduces feeding time in large carnivores. Proc R Soc B 2017;284:20170433.
- [40] Suraci JP, Smith JA, Clinchy M, Zanette LY, Wilmers CC. Humans, but not their dogs, displace pumas from their kills: an experimental approach. Sci Rep 2019;9:1–8.
- [41] Crawford DA, Conner LM, Clinchy M, et al. Prey tells, large herbivores fear the human 'super predator'. Oecologia 2022;198:91–8.
- [42] Beale CM, Monaghan P. Human disturbance: people as predation-free predators? J Appl 2004;41(2):335–43.
- [43] Gill JA, Norris K, Sutherland W. Why behavioural responses may not reflect the population consequences of human disturbance. Biol Cons 2001;97(3):265–8.
- [44] Creel S, Christianson D. Relationships between direct predation and risk effects. Trends Ecol Evol 2008;23(4):194–201.
- [45] Liu W. Criterion of hopf bifurcations without using eigenvalues. J Math Anal Appl 1994;182:250–6.
- [46] Saltelli A. Making best use of model evaluations to compute sensitivity indices. Comput Phys Comm 2002;145(2):280–97.
- [47] Saltelli A, Tarantola S, Chan K. A quantitative model-independent method for global sensitivity analysis of model output. Technometrics 1999;41:39–56.
- [48] Sobol I. Sensitivity estimates for nonlinear mathematical models. Math Model Comput Exp 1993;4:407–14.
- [49] Antoci A, Borghesi S, Russu P. Biodiversity and economic growth: Trade-offs between stabilization of the ecological system and preservation of natural dynamics. Ecol Model 2005;189:333–46.
- [50] Clark C. Mathematical bioeconomics: the optimal management resources. John Wiley & Sons; 1976.

- [51] Clark C, Charles A, Beddington J, Mangel M. Optimal capacity decisions in a developing fishery. *Mar Resour Econ* 1985;2(1):25–53.
- [52] Srinivasu P. Bioeconomics of a renewable resource in presence of a predator. *Nonlinear Anal Real World Appl* 2001;2(4):497–506.
- [53] Gupta R, Malay B, Peeyush C. Period doubling cascades of prey-predator model with nonlinear harvesting and control of over exploitation through taxation. *Commun Nonlinear Sci Numer Simul* 2014;19(7):2382–405.
- [54] Jansen M. Analysis of variance designs for model output. *Comput Phys Comm* 1999;117:35–43.